Step-by-Step Calibration of the Option Pricing & Rates Models

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Introduction

In this short pdf, I am breaking down the calibration of option pricing models and interest rate models step by step, focusing on clear and practical methods for understanding and implementing these processes. We'll cover key models such as the Black-Scholes model, LV model, Vasicek model, CIR and HW Model.

1. Data Collection

- Market Prices of Options (C_{market}): These are the observed prices of options traded in the market. We collect these because they represent the "true" value of the options under current market conditions.
- Underlying Asset Price (S_0): This is the current price of the stock or asset on which the option is based.
- Strike Price (K): This is the price at which the option can be exercised.
- Time to Maturity (T): The remaining time until the option expires.
- Risk-Free Interest Rate (r): The continuously compounded risk-free rate, typically derived from government bonds.

2. Parameter to Calibrate

- Volatility (σ): This is the parameter we need to calibrate. It represents the standard deviation of the asset's returns and is crucial for pricing options.

3. Initial Guess for Volatility

Start with an initial guess for volatility. Let's assume $\sigma_0 = 0.2$ (20%).

4. Black-Scholes Option Pricing Formula

The theoretical price of a European call option under the Black-Scholes model is given by:

$$C = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

where:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and Φ is the cumulative distribution function of the standard normal distribution.

5. Objective Function

The objective function measures the error between market prices and theoretical prices. We aim to minimize this error.

$$\text{Error} = \sum_{i=1}^{n} \left(C_{\text{market},i} - C_{\text{model},i}(\sigma)\right)^2$$

where $C_{\text{model},i}(\sigma)$ is the price calculated using the Black-Scholes formula with volatility σ.

Illustrative Example

Let's say we have the following data for a European call option:

- Market Price of the Option (C_{market}) = \$8
- Underlying Asset Price (S_0) = \$100
- Strike Price (K) = \$100
- Time to Maturity (T) = 1 year
- Risk-Free Interest Rate (r) = 5% (0.05)

Initial Guess

Let's start with an initial guess for volatility $\sigma_0 = 0.2$.

Calculating C_{model}

Compute d_1 and d_2:

$$d_1 = \frac{\ln(100/100) + (0.05 + 0.2^2/2) \cdot 1}{0.2\sqrt{1}} = \frac{0 + 0.07}{0.2} = 0.35$$

$$d_2 = d_1 - 0.2 \cdot \sqrt{1} = 0.35 - 0.2 = 0.15$$

Compute $\Phi(d_1)$ and $\Phi(d_2)$: Using standard normal distribution tables or a calculator:

$$\Phi(0.35) \approx 0.6368$$

$$\Phi(0.15) \approx 0.5596$$

Compute the theoretical option price:

$$C_{\text{model}} = 100 \cdot 0.6368 - 100 \cdot e^{-0.05 \cdot 1} \cdot 0.5596$$

$$C_{\text{model}} = 63.68 - 53.14 = 10.54$$

Error Calculation

$$\text{Error} = (8 - 10.54)^2 = (-2.54)^2 = 6.45$$

Update Volatility

Since the model price (10.54) is higher than the market price (8), we need to reduce the volatility.

New Guess: Let's try $\sigma_1 = 0.15$.

Compute d_1 and d_2 for new σ:

$$d_1 = \frac{\ln(100/100) + (0.05 + 0.15^2/2) \cdot 1}{0.15\sqrt{1}} = \frac{0 + 0.0575}{0.15} = 0.3833$$

$$d_2 = d_1 - 0.15 \cdot \sqrt{1} = 0.3833 - 0.15 = 0.2333$$

Compute $\Phi(d_1)$ and $\Phi(d_2)$ for new σ:

$$\Phi(0.3833) \approx 0.6494$$

$$\Phi(0.2333) \approx 0.5928$$

Compute the theoretical option price for new σ:

$$C_{\text{model}} = 100 \cdot 0.6494 - 100 \cdot e^{-0.05 \cdot 1} \cdot 0.5928$$

$$C_{\text{model}} = 64.94 - 56.37 = 8.57$$

Error Calculation

$$\text{Error} = (8 - 8.57)^2 = (-0.57)^2 = 0.32$$

The error has reduced significantly. We can continue this process, adjusting σ iteratively until the error is minimized.

Final Iteration (if necessary)

If $\sigma = 0.15$ still doesn't provide the desired accuracy, we can adjust it slightly up or down and repeat the calculations. Typically, using numerical optimization methods (like the Newton-Raphson method) can automate this process and converge more quickly to the optimal σ.

Newton-Raphson Method for Calibrating the Black-Scholes Model

The Newton-Raphson method is an iterative technique used for finding the roots of a real-valued function. In the context of calibrating the Black-Scholes model, we use it to minimize the error between the market price and the model price of an option. Here, we'll use the Newton-Raphson method to find the volatility (σ) that minimizes the objective function.

Steps to Apply Newton-Raphson Method

1. Define the Objective Function and Its Derivative

Let $f(\sigma)$ be the difference between the market price C_{market} and the Black-Scholes model price $C_{\text{model}}(\sigma)$:

$$f(\sigma) = C_{\text{model}}(\sigma) - C_{\text{market}}$$

The derivative of $f(\sigma)$ with respect to σ is needed for the Newton-Raphson method. This is denoted as $f'(\sigma)$.

2. Black-Scholes Model Price and Its Sensitivity (Vega)

The Black-Scholes price for a European call option is:

$$C_{\text{model}}(\sigma) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

The vega (sensitivity of the option price with respect to volatility) is:

$$\nu = S_0\sqrt{T}\phi(d_1)$$

where ϕ is the probability density function of the standard normal distribution:

$$\phi(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$$

3. Newton-Raphson Iteration

The Newton-Raphson update formula is:

$$\sigma_{n+1} = \sigma_n - \frac{f(\sigma_n)}{f'(\sigma_n)}$$

Here, $f(\sigma_n)$ is the difference between the model price and the market price, and $f'(\sigma_n)$ is the vega.

Example Calculation

Let's go through a detailed example with the following data:

- Market Price of the Option (C_{market}) = \$8
- Underlying Asset Price (S_0) = \$100
- Strike Price (K) = \$100
- Time to Maturity (T) = 1 year
- Risk-Free Interest Rate (r) = 5% (0.05)

Initial Guess

Start with an initial guess $\sigma_0 = 0.2$.

Iteration 1

Compute d_1 and d_2 for σ_0:

$$d_1 = \frac{\ln(100/100) + (0.05 + 0.2^2/2) \cdot 1}{0.2\sqrt{1}} = \frac{0 + 0.07}{0.2} = 0.35$$

$$d_2 = d_1 - 0.2\sqrt{1} = 0.35 - 0.2 = 0.15$$

Compute $\Phi(d_1)$ and $\Phi(d_2)$:

$$\Phi(0.35) \approx 0.6368$$
$$\Phi(0.15) \approx 0.5596$$

Compute the theoretical option price:

$$C_{\text{model}} = 100 \cdot 0.6368 - 100 \cdot e^{-0.05\cdot 1} \cdot 0.5596$$
$$C_{\text{model}} = 63.68 - 53.14 = 10.54$$

Compute Vega (ν):

$$\phi(d_1) = \frac{1}{\sqrt{2\pi}} e^{-0.35^2/2} \approx 0.3752$$
$$\nu = 100 \cdot \sqrt{1} \cdot 0.3752 = 37.52$$

Compute the difference and update σ:

$$f(\sigma_0) = 10.54 - 8 = 2.54$$
$$\sigma_1 = 0.2 - \frac{2.54}{37.52} \approx 0.2 - 0.0677 = 0.1323$$

Iteration 2

Compute d_1 and d_2 for σ_1:

$$d_1 = \frac{\ln(100/100) + (0.05 + 0.1323^2/2) \cdot 1}{0.1323\sqrt{1}} = \frac{0 + 0.0283}{0.1323} \approx 0.2137$$

$$d_2 = d_1 - 0.1323\sqrt{1} = 0.2137 - 0.1323 = 0.0814$$

Compute $\Phi(d_1)$ and $\Phi(d_2)$:

$$\Phi(0.2137) \approx 0.5853$$
$$\Phi(0.0814) \approx 0.5324$$

Compute the theoretical option price:

$$C_{\text{model}} = 100 \cdot 0.5853 - 100 \cdot e^{-0.05\cdot 1} \cdot 0.5324$$
$$C_{\text{model}} = 58.53 - 50.61 = 7.92$$

Compute Vega (ν):

$$\phi(d_1) = \frac{1}{\sqrt{2\pi}} e^{-0.2137^2/2} \approx 0.3910$$
$$\nu = 100 \cdot \sqrt{1} \cdot 0.3910 = 39.10$$

Compute the difference and update σ:

$$f(\sigma_1) = 7.92 - 8 = -0.08$$
$$\sigma_2 = 0.1323 - \frac{-0.08}{39.10} \approx 0.1323 + 0.0020 = 0.1343$$

Iteration 3 (Optional)

Continue iterating in the same manner until the difference $f(\sigma_n)$ is close to zero, indicating that the theoretical price matches the market price.

Summary

1. Define the objective function $f(\sigma)$ and its derivative (vega).
2. Make an initial guess for volatility.
3. Iteratively update the volatility using the Newton-Raphson formula.
4. Stop when the error between the market price and the model price is minimized.

The local volatility model extends the Black-Scholes model by allowing the volatility to be a function of both the underlying asset price S and time t, i.e., $\sigma = \sigma(S,t)$. This model captures the observed market phenomenon where implied volatilities vary with strike prices and maturities.

Steps for Calibrating the Local Volatility Model

Data Collection

- **Market Data**: Collect option prices for a range of strike prices and maturities.
- **Implied Volatilities**: Convert market prices to implied volatilities using the Black-Scholes formula.

Dupire's Equation

Dupire's local volatility function $\sigma_{LV}(S,t)$ can be derived from the market prices of European call options $C(K,T)$:

$$\sigma_{LV}^2(K,T) = \frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$$

Here, $C(K,T)$ is the market price of a European call option with strike K and maturity T.

Finite Differences Method

To compute σ_{LV}, we need the partial derivatives of the option prices:

$$\frac{\partial C}{\partial T}, \quad \frac{\partial C}{\partial K}, \quad \frac{\partial^2 C}{\partial K^2}$$

Example Calculation

Let's go through an example step by step using simple finite differences.

Market Data

Option Prices: We have the following market prices for a given maturity T:

$$C(90,T) = 12, \quad C(100,T) = 8, \quad C(110,T) = 5$$

Risk-Free Rate: $r = 5\% = 0.05$

Finite Differences

First Partial Derivative with Respect to Strike Price K:

$$\frac{\partial C}{\partial K} \approx \frac{C(K+\Delta K) - C(K-\Delta K)}{2\Delta K}$$

Let's choose $\Delta K = 10$:

$$\left.\frac{\partial C}{\partial K}\right|_{K=100} \approx \frac{C(110,T) - C(90,T)}{2 \cdot 10} = \frac{5-12}{20} = -0.35$$

Second Partial Derivative with Respect to Strike Price K:

$$\frac{\partial^2 C}{\partial K^2} \approx \frac{C(K+\Delta K) - 2C(K) + C(K-\Delta K)}{(\Delta K)^2}$$

Using the same $\Delta K = 10$:

$$\left.\frac{\partial^2 C}{\partial K^2}\right|_{K=100} \approx \frac{C(110,T) - 2C(100,T) + C(90,T)}{10^2} = \frac{5 - 2 \cdot 8 + 12}{100} = \frac{1}{100} = 0.01$$

First Partial Derivative with Respect to Maturity T:
Assume we have the prices for slightly different maturities T and $T + \Delta T$. For simplicity, assume:

$$C(100,T) = 8, \quad C(100, T+\Delta T) = 7.9$$

Let $\Delta T = 0.01$ (1 day):

$$\frac{\partial C}{\partial T} \approx \frac{C(100,T+\Delta T) - C(100,T)}{\Delta T} = \frac{7.9-8}{0.01} = -10$$

Compute Local Volatility

Using Dupire's equation:

$$\sigma_{LV}^2(100,T) = \frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$$

Plugging in the values:

$$\sigma_{LV}^2(100,T) = \frac{-10 + 0.05 \cdot 100 \cdot (-0.35)}{\frac{1}{2} \cdot 100^2 \cdot 0.01}$$

$$\sigma_{LV}^2(100,T) = \frac{-10 - 1.75}{50} = \frac{-11.75}{50} = -0.235$$

Since volatility cannot be negative, an error might be present in the initial assumptions or calculations. Typically, a larger data set or smoothing techniques are employed to reduce such discrepancies.

Iterative Process and Updates

The process outlined can be iteratively refined:

- **Initial Guess**: Start with an initial guess for $\sigma_{LV}(K,T)$.
- **Compute Partial Derivatives**: Using market data, compute the partial derivatives using finite differences.
- **Update Volatility**: Apply Dupire's equation to update the local volatility surface.
- **Repeat**: Iterate the process to refine the local volatility estimates.

Vasicek Model Calibration

The Vasicek model describes the evolution of the short-term interest rate $r(t)$ with the following stochastic differential equation (SDE):

$$dr(t) = \kappa(\theta - r(t))\,dt + \sigma\,dW(t)$$

where:

- κ: Speed of mean reversion (how quickly the rate reverts to the long-term mean)
- θ: Long-term mean level of the rate
- σ: Volatility of the rate
- $W(t)$: Standard Brownian motion

Calibration Steps

1. Collect Market Data

To calibrate the Vasicek model, you need market data, such as:

- Zero-coupon bond prices: Prices of bonds with different maturities.
- Interest rate swaps: Market prices of interest rate swaps with various tenors.

2. Calculate the Model-Implied Prices

The Vasicek model price of a zero-coupon bond $P(t, T)$ maturing at time T can be expressed as:

$$P(t,T) = A(t,T)\exp\left(-B(t,T)\cdot r(t)\right)$$

where $A(t, T)$ and $B(t, T)$ are functions that depend on the model parameters. For the Vasicek model:

$$B(t,T) = \frac{1-\exp(-\kappa(T-t))}{\kappa}$$

$$A(t,T) = \exp\left(\theta\frac{\sigma^2}{2\kappa^2}\left[(T-t) - \frac{2\exp(-\kappa(T-t))}{2\kappa}\right]\right)$$

3. Objective Function for Calibration

Define the objective function to minimize the difference between the model-implied prices and the market prices. For zero-coupon bonds:

$$F(\kappa,\theta,\sigma) = \sum_{i=1}^{N}\left(P_{\text{market},i} - P_{\text{model},i}\right)^2$$

where:

- $P_{\text{market},i}$: Market price of the i-th bond.
- $P_{\text{model},i}$: Model price of the i-th bond, computed using the Vasicek model.

4. Minimize the Objective Function

Use optimization techniques to find the parameters κ, θ, and σ that minimize the objective function. Common methods include:

- Gradient Descent: Iteratively adjust parameters in the direction that reduces the objective function.
- Newton-Raphson Method: Uses second-order information (the Hessian matrix) to find the minimum.

Detailed Example: Calibration of Vasicek Model

Market Data

Suppose you have the following market prices of zero-coupon bonds with maturities T_i:

- Bond 1 (Maturity = 1 year): $P_{\text{market},1} = 0.98$
- Bond 2 (Maturity = 2 years): $P_{\text{market},2} = 0.95$
- Bond 3 (Maturity = 3 years): $P_{\text{market},3} = 0.92$

Initial Parameter Guesses

Let's start with the following initial guesses:

$$\kappa = 0.5, \quad \theta = 0.03, \quad \sigma = 0.015$$

Compute Model Prices

For each maturity, compute $B(t, T)$ and $A(t, T)$ using the formulas provided above. Assume $r(t)$ is the current short-term interest rate (say 0.02).

Example Calculation for Bond 1 (Maturity = 1 year):

$$T = 1$$

$$B(t, 1) = \frac{1 - \exp(-0.5 \times 1)}{0.5} \approx 0.632$$

$$A(t, 1) = \exp\left(\theta \frac{\sigma^2}{2\kappa^2}\left[(T - t) - \frac{2\exp(-\kappa(T - t))}{2\kappa}\right]\right) \approx 0.982$$

$$P_{\text{model},1} = 0.982 \times \exp(-0.632 \times 0.02) \approx 0.98$$

Repeat this process for Bonds 2 and 3.

Minimize the Objective Function

Calculate the Objective Function:

$$F(\kappa, \theta, \sigma) = (0.98 - 0.98)^2 + (0.95 - \text{Model Price 2})^2 + (0.92 - \text{Model Price 3})^2$$

Optimize: Adjust κ, θ, and σ to minimize F using an optimization algorithm.

Newton-Raphson Method

To use the Newton-Raphson method, we need the gradient of the objective function with respect to the parameters. Let's focus on $\frac{\partial F}{\partial \theta}$.

Compute the Partial Derivatives

The gradient ∇F with respect to each parameter κ, θ, and σ is needed. For $\frac{\partial F}{\partial \theta}$:

$$\frac{\partial F}{\partial \theta} = -2 \sum_{i=1}^{N} (P_{\text{market},i} - P_{\text{model},i}) \cdot \frac{\partial P_{\text{model},i}}{\partial \theta}$$

For the model price:

$$P(t, T) = A(t, T)\exp(-B(t, T) \cdot r(t))$$

$$\frac{\partial P_{\text{model},i}}{\partial \theta} = \left(\frac{\partial A(t, T)}{\partial \theta}\right)\exp(-B(t, T) \cdot r(t))$$

$$\frac{\partial A(t, T)}{\partial \theta} = A(t, T) \cdot \frac{\sigma^2}{2\kappa^2}\left[(T - t) - \frac{2\exp(-\kappa(T - t))}{2\kappa}\right]$$

Example Calculation

For Bond 1:

$$T = 1$$

$$P_{\text{market},1} = 0.98, \quad P_{\text{model},1} = 0.97$$

$$\frac{\partial A(t,T)}{\partial \theta} = A(t,T) \cdot 0.000225 \, [1 - \exp(-0.5)]$$

Compute $\frac{\partial P_{\text{model},1}}{\partial \theta}$:

$$\frac{\partial P_{\text{model},1}}{\partial \theta} = \left(\frac{\partial A(t,T)}{\partial \theta} \right) \exp(-B(t,T) \cdot r(t))$$

Compute the Gradient:

$$\frac{\partial F}{\partial \theta} = -2(0.98 - 0.97) \cdot \frac{\partial P_{\text{model},1}}{\partial \theta}$$

Summary

- Collect Market Data: Zero-coupon bond prices or interest rate swaps.
- Compute Model Prices: Use the Vasicek model formulas for bonds or swaps.
- Define Objective Function: Measure the difference between market and model prices.
- Optimize Parameters: Use numerical methods to find the parameters that minimize the objective function.

1 CIR Model

The CIR model describes the short-term interest rate $r(t)$ with the following stochastic differential equation (SDE):

$$dr(t) = \kappa(\theta - r(t))\, dt + \sigma \sqrt{r(t)}\, dW(t) \tag{1}$$

Where:

- κ: Speed of mean reversion
- θ: Long-term mean level of the rate
- σ: Volatility of the rate
- $W(t)$: Standard Brownian motion

2 Calibration Steps

2.1 Collect Market Data

As with the Vasicek model, you need market data such as:

- Zero-coupon bond prices: Prices of bonds with different maturities.
- Interest rate swaps: Prices of swaps with various tenors.

2.2 Compute the Model-Implied Prices

For the CIR model, the price of a zero-coupon bond $P(t,T)$ maturing at time T is given by:

$$P(t,T) = A(t,T) \exp\left(-B(t,T) \cdot r(t)\right) \tag{2}$$

Where $A(t,T)$ and $B(t,T)$ are functions that depend on the model parameters. For the CIR model:

$$B(t,T) = \frac{2\left(\exp(\kappa(T-t)) - 1\right)}{2\kappa + (\sigma^2/\theta)\left(\exp(\kappa(T-t)) - 1\right)} \tag{3}$$

$$A(t,T) = \exp\left(\frac{2\kappa\theta}{\sigma^2} \left[B(t,T) - (T-t) - \frac{\sigma^2}{2\kappa}\left(B(t,T) - (T-t)\right) \right] \right) \tag{4}$$

2.3 Objective Function for Calibration

Define the objective function to minimize the difference between the model-implied prices and the market prices:

$$F(\kappa, \theta, \sigma) = \sum_{i=1}^{N} \left(P_{\text{market},i} - P_{\text{model},i}\right)^2 \tag{5}$$

Where $P_{\text{model},i}$ is the price computed using the CIR model and $P_{\text{market},i}$ is the market price.

2.4 Minimize the Objective Function

Use optimization techniques to find the parameters κ, θ, and σ that minimize the objective function.

3 Detailed Example: Calibration of CIR Model

3.1 Market Data

Suppose you have the following market prices of zero-coupon bonds with maturities T_i:

- Bond 1 (Maturity = 1 year): $P_{\text{market},1} = 0.98$
- Bond 2 (Maturity = 2 years): $P_{\text{market},2} = 0.95$
- Bond 3 (Maturity = 3 years): $P_{\text{market},3} = 0.92$

3.2 Initial Parameter Guesses

Let's start with the following initial guesses:

- $\kappa = 0.5$
- $\theta = 0.03$
- $\sigma = 0.015$

3.3 Compute Model Prices

Compute $B(t,T)$ and $A(t,T)$:

For each maturity, compute $B(t,T)$ and $A(t,T)$ using the formulas provided above. Assume $r(t)$ is the current short-term interest rate (say 0.02).

Compute $P_{\text{model},i}$:

Use the computed $A(t,T)$ and $B(t,T)$ to get the model prices.

Example Calculation: For Bond 1:

- $T = 1$

Compute $B(t,1)$ and $A(t,1)$:

$$B(t,1) = \frac{2\left(\exp(0.5 \times 1) - 1\right)}{2 \times 0.5 + \left(\frac{0.015^2}{0.03}\right)\left(\exp(0.5 \times 1) - 1\right)} \tag{6}$$

$$A(t,1) = \exp\left(\frac{2 \times 0.5 \times 0.03}{0.015^2}\left[B(t,1) - (1-0) - \frac{0.015^2}{2 \times 0.5 \times 0.03}\left(B(t,1) - (1-0)\right)\right]\right) \tag{7}$$

Compute these values to find $P_{\text{model},1}$.

3.4 Minimize the Objective Function

Calculate the Objective Function:

$$F(\kappa, \theta, \sigma) = (0.98 - \text{Model Price 1})^2 + (0.95 - \text{Model Price 2})^2 + (0.92 - \text{Model Price 3})^2 \tag{8}$$

Optimize:

Adjust κ, θ, and σ to minimize F using an optimization algorithm.

3.5 Gradient Calculation for Newton-Raphson

To use the Newton-Raphson method, we need the gradient of the objective function with respect to κ, θ, and σ. For $\frac{\partial F}{\partial \theta}$, the gradient is:

$$\frac{\partial F}{\partial \theta} = -2 \sum_{i=1}^{N} \left(P_{\text{market},i} - P_{\text{model},i} \right) \cdot \frac{\partial P_{\text{model},i}}{\partial \theta} \tag{9}$$

Where $\frac{\partial P_{\text{model},i}}{\partial \theta}$ can be derived from:

$$\frac{\partial P_{\text{model},i}}{\partial \theta} = \frac{\partial A(t,T)}{\partial \theta} \cdot \exp\left(-B(t,T) \cdot r(t)\right) \tag{10}$$

To compute $\frac{\partial A(t,T)}{\partial \theta}$:

$$\frac{\partial A(t,T)}{\partial \theta} = A(t,T) \cdot \frac{2\sigma^2}{\theta^2} \left[B(t,T) - (T-t) \right] \tag{11}$$

Hull-White Model

The Hull-White model describes the short-term interest rate $r(t)$ with the following stochastic differential equation (SDE):

$$dr(t) = \theta(t)\, dt + \sigma(t)\, dW(t)$$

where:

- $\theta(t)$ is the deterministic drift term.
- $\sigma(t)$ is the time-dependent volatility term.
- $W(t)$ is a standard Brownian motion.

For the Hull-White model, the term structure of the interest rate is given by:

$$P(t,T) = A(t,T) \exp(-B(t,T) \cdot r(t))$$

where $A(t,T)$ and $B(t,T)$ are functions of the model parameters. Specifically:

$$B(t,T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha}$$

$$A(t,T) = \exp\left[\frac{\sigma^2}{2\alpha^2} \left(B(t,T)^2 - (T-t) \right) \right]$$

where:

- α is the mean-reversion speed.
- σ is the volatility of the short-term interest rate.

Calibration Steps

Collect Market Data

To calibrate the Hull-White model, you need:

- Zero-Coupon Bond Prices: Prices of bonds with different maturities.
- Interest Rate Swaps: Prices of swaps with various tenors.

Compute Model-Implied Prices

For zero-coupon bonds, use the Hull-White model formulas to compute the model price:

$$P_{\text{model}}(t,T) = A(t,T) \exp(-B(t,T) \cdot r(t))$$

Define the Objective Function

The objective function F to minimize is the sum of squared differences between market prices and model prices:

$$F(\alpha, \sigma) = \sum_{i=1}^{N} (P_{\text{market},i} - P_{\text{model},i})^2$$

Optimize Parameters

Use optimization techniques to find the parameters α and σ that minimize the objective function.

Detailed Example: Calibration of Hull-White Model

Market Data

Suppose you have the following market prices of zero-coupon bonds with maturities T_i:

- Bond 1 (Maturity = 1 year): $P_{\text{market},1} = 0.98$
- Bond 2 (Maturity = 2 years): $P_{\text{market},2} = 0.95$
- Bond 3 (Maturity = 3 years): $P_{\text{market},3} = 0.92$

Initial Parameter Guesses

Let's start with the following initial guesses:

- $\alpha = 0.5$
- $\sigma = 0.015$

Compute Model Prices

Compute $B(t, T)$ and $A(t, T)$:
For Bond 1 (Maturity = 1 year):

$$B(t, 1) = \frac{1 - e^{-0.5 \times 1}}{0.5} \approx 0.632$$

$$A(t, 1) = \exp\left[\frac{0.015^2}{2 \times 0.5^2}\left(0.632^2 - 1\right)\right]$$

$$A(t, 1) \approx \exp\left[\frac{0.000225}{0.5}(0.399 - 1)\right] \approx \exp(-0.000045) \approx 0.999955$$

So, the model price for Bond 1:

$$P_{\text{model},1} = 0.999955 \times \exp(-0.632 \times r(t))$$

Compute Prices for Other Bonds

Repeat the process for Bonds 2 and 3 using their maturities.

Define and Minimize Objective Function

Calculate the Objective Function:
Define $F(\alpha, \sigma)$ as:

$$F(\alpha, \sigma) = (0.98 - P_{\text{model},1})^2 + (0.95 - P_{\text{model},2})^2 + (0.92 - P_{\text{model},3})^2$$

Optimize Parameters

Use numerical optimization techniques (e.g., gradient descent, Newton-Raphson) to adjust α and σ to minimize F.

Gradient Calculation for Newton-Raphson

To use the Newton-Raphson method, we need the gradients of F with respect to α and σ. For $\frac{\partial F}{\partial \alpha}$, the gradient is:

$$\frac{\partial F}{\partial \alpha} = -2 \sum_{i=1}^{N} (P_{\text{market},i} - P_{\text{model},i}) \cdot \frac{\partial P_{\text{model},i}}{\partial \alpha}$$

where $\frac{\partial P_{\text{model},i}}{\partial \alpha}$ can be derived from:

$$\frac{\partial P_{\text{model},i}}{\partial \alpha} = \left(\frac{\partial A(t,T)}{\partial \alpha} \cdot \exp(-B(t,T) \cdot r(t)) \right)$$

Similarly, compute the gradient with respect to σ:

$$\frac{\partial F}{\partial \sigma} = -2 \sum_{i=1}^{N} (P_{\text{market},i} - P_{\text{model},i}) \cdot \frac{\partial P_{\text{model},i}}{\partial \sigma}$$

where $\frac{\partial P_{\text{model},i}}{\partial \sigma}$ can be derived from:

$$\frac{\partial P_{\text{model},i}}{\partial \sigma} = \left(\frac{\partial A(t,T)}{\partial \sigma} \cdot \exp(-B(t,T) \cdot r(t)) \right)$$

Summary

1. Collect Market Data: Zero-coupon bond prices or interest rate swaps.
2. Compute Model Prices: Use Hull-White model formulas.
3. Define Objective Function: Measure the difference between market and model prices.
4. Compute Gradients: Derive partial derivatives for Newton-Raphson.
5. Optimize Parameters: Use numerical methods to minimize the objective function.