Quanto CDS

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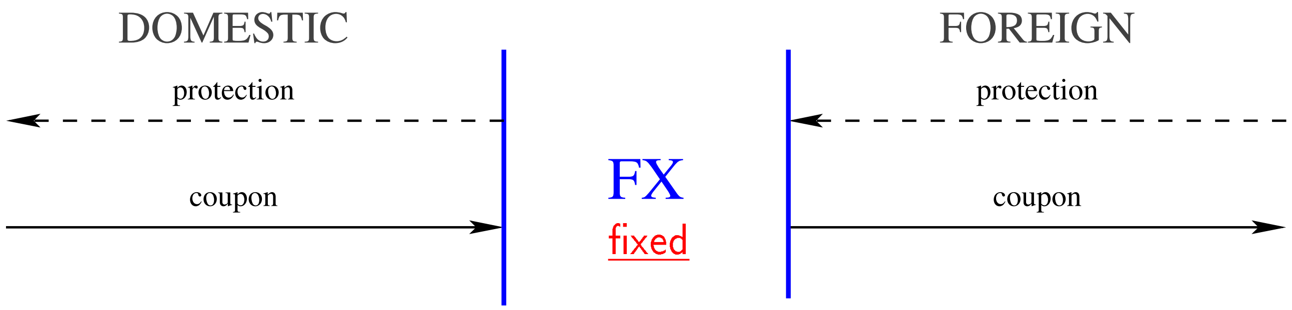
What is a Quanto CDS?

- Credit default swaps are widely used to hedge credit risk.
- For practically all reference entities, there is, at best, a liquid CDS market only *in one currency*.
- Credit risk hedging needs often arise *for other currencies*.
- A compound (back-to-back) package of a credit default swap in the liquid *and* the foreign currency is known as a Quanto CDS.



- Note that the notionals in both currencies are fixed upfront, i.e., *the effective FX rate is fixed at inception*.

- Protection on Boeing in EUR.
- Protection on Carrefour in USD.
- Protection on Sovereign Belgium in USD.
- Protection on GazProm in USD.

Most CDS are quoted in USD[1].

Quanto CDS tend to be quoted against USD.

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Subject: Italy Curve USD & EUR **** 250mm€ traded so far****

91)☆ 92) Move 94) Tags

MATURITY	USD	EURO	Quanto
1Y	45/55	25/45	
2Y	65/80	47/67	10/25
3Y	92/100	53/73	25/35
4Y	115/120	74/82	
5Y	129/132	87/94	37/42
7Y	135/139	93/100	
10y	142/147	99/106	38/44

It tends to be left up to the reader to guess the sign of the Quanto quote.

[1] About 60% by notional according to DTC [CDS10].

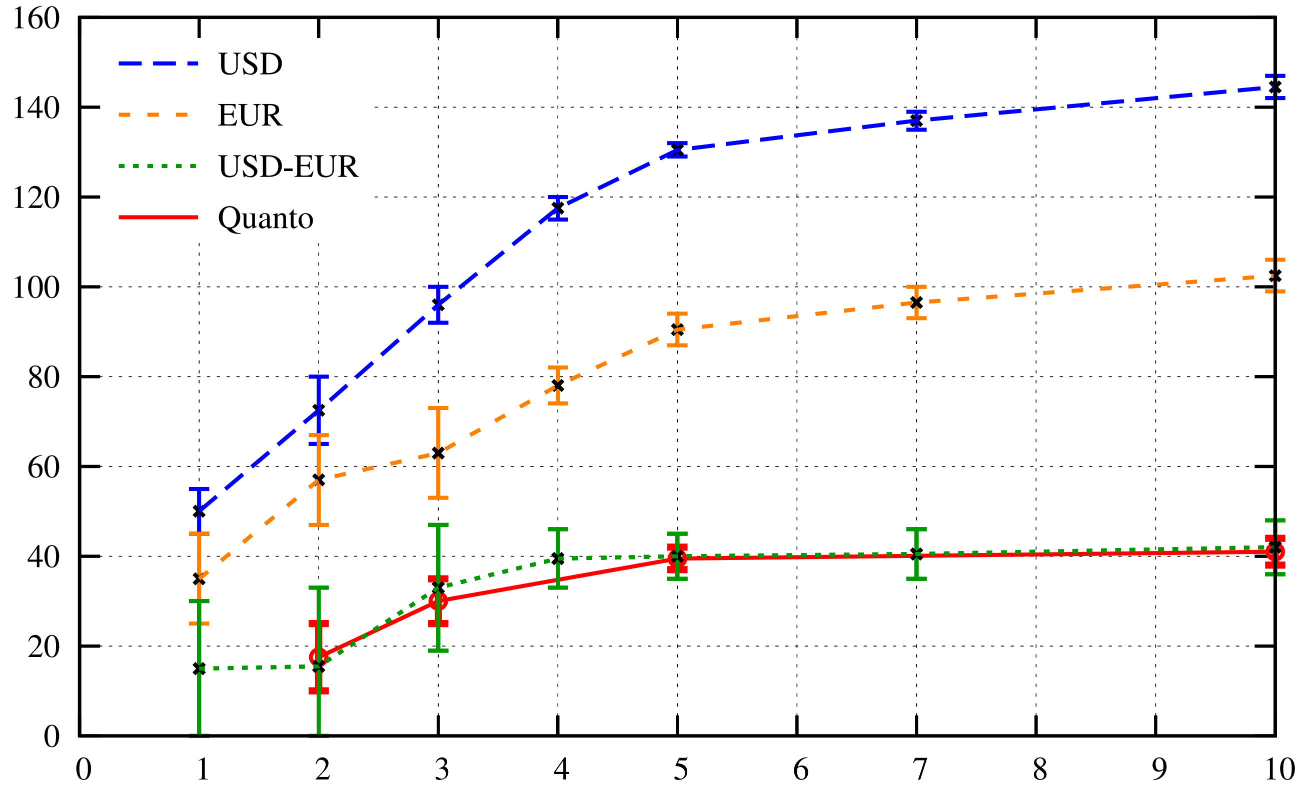

Figure 1: USD, EUR, and Quanto spread quotes on sovereign Italy on April 13th 2011.

Quanto CDS also arise in the hedges for:-

- First-to-default swaps and
- Synthetic collateralised debt obligation tranches

  on baskets, e.g., of a mix of European and North American names.

- CVA protection.

The main risks in a Quanto CDS

- Default
- Spread moves
- FX moves
- The dominant risk in FX moves is the resulting mismatch of the protection leg notionals.

Example

You enter into the following (naked) Quanto CDS:-

USDRUB	Long	Protection	Short	Protection	Net	Protection
30	300	MM RUB	10	MM USD	0	MM USD

FX moves to 40:

USDRUB	Long	Protection	Short	Protection	Net	Protection
40	300	MM RUB	10	MM USD	-2.5	MM USD

FX moves to 20:

USDRUB	Long	Protection	Short	Protection	Net	Protection
20	300	MM RUB	10	MM USD	+5	MM USD

The key drivers for Quanto CDS spreads are therefore:-

- Cost of rehedging to re-balance the protection leg notionals.
- Rapid FX moves happening *just as the entity descends into default*.

It appears that market prices for Quanto CDS are dominated by the latter [Mos10] (at least) for sovereign and key corporate debt (which tends to be the only market for Quanto CDS).

Valuation based on conventional (re-)hedging arguments seems to be precarious [Pol12]:

Quanto CDS, the widow-maker

Posted by **Lisa Pollack** on Mar 08 16:15.

In the space of less than a year, the headline in the International Financing Review went from "Quanto CDS flows return" to "Dealers hope for the death of quanto CDS".

[...]

A year ago, IFR reported that while flows in the market had returned, dealers had recently taken a bath on their quanto positions on Greece (emphasis ours):

> *The basis between Greek CDS with payments in US dollars (the standard contract) and in euros (the quanto) leapt to 100bp, having traded as close as 1bp in 2006. As sovereign risk contagion spread to other peripheral European countries,* ***many desks are thought to have incurred losses on their quanto positions****, which became an ever more important risk to monitor.*

[...]

Oh, and it's been an absolute pain to hedge the FX risk in these things:

> *The one-sided nature of dealer books has led to many losses in the past, and hedging these positions has not got any easier. As one trader points out,* ***no one can accurately conceptualise how the euro will react if a major country like Germany or Italy defaults. As a result, any risk management of these positions is very much a finger-in-the-air affair.***

Hence sentiments like this:

> *"It's a totally illiquid market and it will just roll off the books.* ***The quanto has no value.*** *Everyone got put in awful positions way back when and* ***now they just want it to go away****," said a senior sovereign trader at a US bank.*

Ouch.

> *"****Quanto CDS has been the widow-maker for many desks.*** *This asset class is completely driven by technicals, which means that taking advantage of apparent fundamental mis-valuations can be a very expensive exercise as the exit options are limited," said Peter Duenas-Brckovich, global co-head of credit flow trading at Nomura.*

Double ouch.

Right, that's that then, we suppose...

Jump at default?

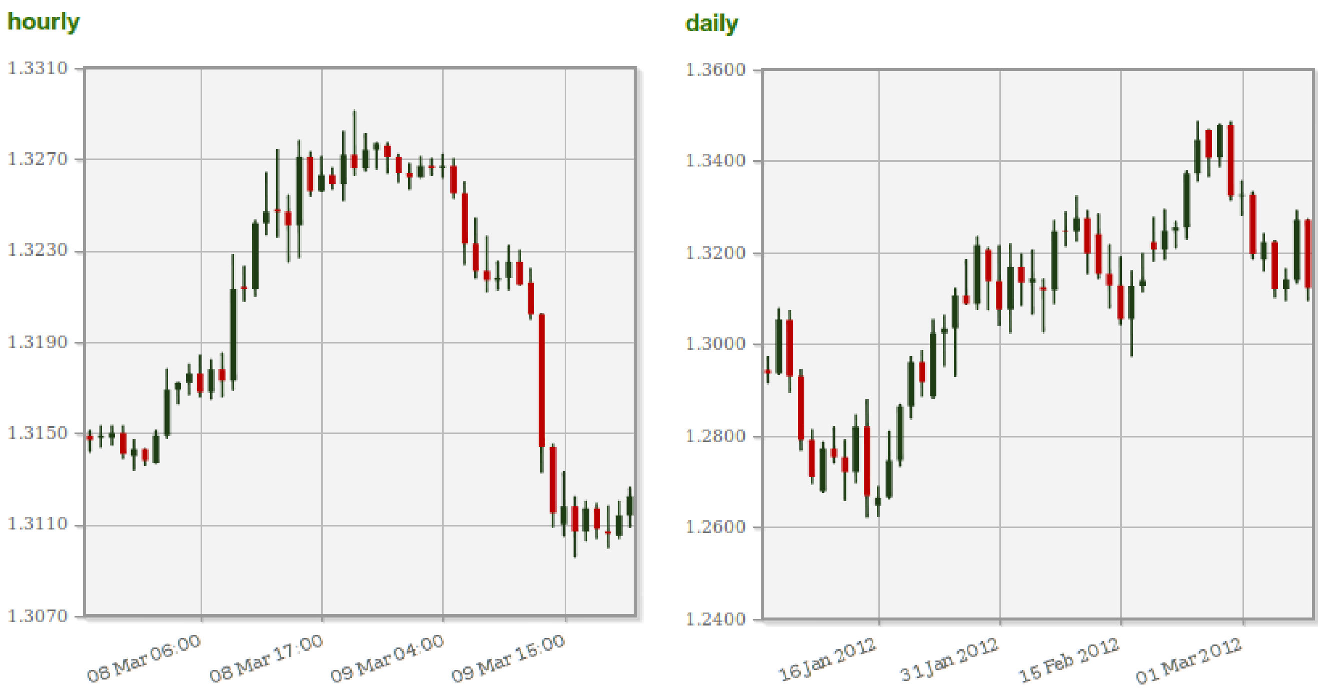

Figure 2: EURUSD on and during the period leading up to the ISDA 9th of March 2012 announcement of a Credit Default Event for the Hellenic Republic.

In a simplified setting, the *protection* or *benefit* leg of a vanilla credit default swap of final maturity T on a unit notional can be valued according to

$$B(T) = \int_0^T P_\tau^{\mathrm{DOM}}(0) \cdot \mathrm{E}^{\mathcal{M}\left(P_\tau^{\mathrm{DOM}}\right)}\left[(1-R)\cdot \mathrm{d}\left(-\mathrm{e}^{-\Lambda(\tau)}\right)\right] . \qquad (1)$$

$P_\tau^{\mathrm{DOM}}(0)$	:	domestic risk-free τ-maturity zero coupon bond observed at $t=0$
$\mathrm{E}^{\mathcal{M}\left(P_\tau^{\mathrm{DOM}}\right)}[\cdot]$	:	expectation in the measure induced by P_τ^{DOM} (also referred to as the τ-forward measure)
R	:	realized recovery rate at the time of default
$\Lambda(t)$	:	a non-decreasing (stochastic) process that we refer to as the *hazard* process.

For conceptual convenience, one may wish to think of the *hazard* process $\Lambda(t)$ as the time integral of a (stochastic) *hazard rate* process

$$\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s \qquad (2)$$

or, equivalently,

$$B(T) = \int_0^T P_t^{\mathrm{DOM}}(0) \cdot \mathrm{E}^{\mathcal{M}\left(P_t^{\mathrm{DOM}}\right)}\left[(1-R)\cdot \mathrm{e}^{-\Lambda(t)} \cdot \dot{\Lambda}(t)\right] \mathrm{d}t . \qquad (3)$$

Assuming further that the CDS premium is payed as a continuous stream of cash at constant rate s, the premium leg is worth $s \cdot A(T)$ with

$$A(T) = \int_0^T P_t^{\mathrm{DOM}}(0) \cdot \mathrm{E}^{\mathcal{M}\left(P_t^{\mathrm{DOM}}\right)}\left[\mathrm{e}^{-\Lambda(t)}\right] \mathrm{d}t . \qquad (4)$$

When the hazard process is deterministic, i.e., a deterministic function of time, which we denote as $\bar{\Lambda}(t)$, and recovery is assumed to be a constant, the benefit leg is

$$B(T) = (1-R)\cdot\int_0^T P_t^{\mathrm{DOM}}(0)\cdot \mathrm{e}^{-\bar{\Lambda}(t)}\cdot\bar{\lambda}(t)\mathrm{d}t \tag{5}$$

with

$$\bar{\lambda}(t) = \dot{\bar{\Lambda}}(t)\ , \tag{6}$$

and the risky annuity becomes

$$A(T) = \int_0^T P_t^{\mathrm{DOM}}(0)\cdot \mathrm{e}^{-\bar{\Lambda}(t)}\mathrm{d}t\ . \tag{7}$$

For a Quanto CDS, all payments are made in a foreign currency.

Let $Q(t)$ denote the value of one foreign currency unit expressed in the reference credit's domestic currency.

By the fundamental theorem of asset pricing [HP81], the value of the Quanto benefit leg, expressed in units of the foreign (Quanto) currency, is

$$B^{\mathrm{Quanto}}(T) = \int_0^T P_t^{\mathrm{DOM}}(0)\cdot \mathrm{E}^{\mathcal{M}(P_t^{\mathrm{DOM}})}\left[(1-R)\cdot\frac{Q(t)}{Q(0)}\cdot\mathrm{d}\left(-\mathrm{e}^{-\Lambda(t)}\right)\right]\ . \tag{8}$$

Equally, we obtain for the value of the Quanto risky annuity, in terms of foreign currency,

$$A^{\text{Quanto}}(T) = \int_0^T P_t^{\text{DOM}}(0) \cdot \mathsf{E}^{\mathcal{M}(P_t^{\text{DOM}})}\left[\frac{Q(t)}{Q(0)} \cdot \mathrm{e}^{-\Lambda(t)}\right] \mathrm{d}t \,. \tag{9}$$

We can see in these equations that, for the pricing of a Quanto CDS, it suffices to know the joint distribution of

- the hazard process $\Lambda(t)$ and
- the FX rate $Q(t)$.

Black FX + Normal hazard rates

The (domestic) FX process given by

$$Q(T) = \hat{Q}(T) \cdot \mathrm{e}^{-\frac{1}{2}\int_0^T \sigma_Q(t)^2 \mathrm{d}t + \int_0^T \sigma_Q(t) \mathrm{d}W_Q(t)} \tag{10}$$

with $\hat{Q}(T)$ being the T-forward for Q, and

$$\Lambda(T) = \int_0^T \lambda(t)\mathrm{d}t \tag{11}$$

$$\lambda(t) = \hat{\lambda}(t) + x(t) \tag{12}$$

$$\mathrm{d}x(t) = -\kappa \cdot x(t)\mathrm{d}t + \sigma_x^{\text{abs}}(t)\mathrm{d}W_x(t) \tag{13}$$

for the (domestic) hazard process Λ with $x(0) = 0$. The function $\hat{\lambda}(t)$ is a deterministic offset that allows us to calibrate the process such that

$$\mathsf{E}\left[\mathrm{e}^{-\Lambda(T)}\right] = \mathrm{e}^{-\bar{\Lambda}(T)} \tag{14}$$

for some (assumed) given $\bar{\Lambda}(T)$.

- The processes $W_Q(t)$ and $W_x(t)$ are standard Wiener processes with instantaneous correlation

$$\mathrm{E}\left[\mathrm{d}W_Q \cdot \mathrm{d}W_x\right] = \rho \cdot \mathrm{d}t \ . \tag{15}$$

- The hazard rate $\lambda(t)$ is an Ornstein-Uhlenbeck process, and hence
- its integral, the hazard $\Lambda(t)$, is a normal process.
- Using the definitions

$$K(t) := \int_0^t \kappa(s)\mathrm{d}s \ , \qquad \text{and} \qquad \hat{\Lambda}(t) := \int_0^t \hat{\lambda}(s)\mathrm{d}s \tag{16}$$

- we note

$$x(t) = \int_0^t \mathrm{e}^{-[K(t)-K(s)]}\sigma_x^{\mathrm{abs}}(s)\mathrm{d}W_x(s) \tag{17}$$

$$\Lambda(T) = \hat{\Lambda}(T) + \int_{s=0}^T \mathrm{e}^{K(s)} \cdot \int_{t=s}^T \mathrm{e}^{-K(t)}\mathrm{d}t \cdot \sigma_x^{\mathrm{abs}}(s) \cdot \mathrm{d}W_x(s) \, . \tag{18}$$

Evidently, $\Lambda(T)$ is normally distributed with expectation $\hat{\Lambda}(T)$

and variance

$$\mathrm{V}\left[\Lambda(T)\right] = \int_{s=0}^T \mathrm{e}^{2K(s)} \left(\int_{t=s}^T \mathrm{e}^{-K(t)}\mathrm{d}t \cdot \sigma_x^{\mathrm{abs}}(s)\right)^2 \mathrm{d}s \ . \tag{19}$$

For constant κ and σ_x^{abs}, this is

$$\mathrm{V}\left[\Lambda(T)\right] = \frac{{\sigma_x^{\mathrm{abs}}}^2}{\kappa^3}\left[\kappa T - 2\left(1-\mathrm{e}^{-\kappa T}\right) + \tfrac{1}{2}\left(1-\mathrm{e}^{-2\kappa T}\right)\right] \tag{20}$$

$$\approx {\sigma_x^{\mathrm{abs}}}^2 \cdot T^3 \cdot \left[\tfrac{1}{3} - \tfrac{1}{4}\kappa T + \tfrac{7}{60}\kappa^2 T^2 + \mathcal{O}(\kappa^3 T^3)\right] \tag{21}$$

for small κT.

In the foreign measure, (also referred to as *Quanto* measure), the process x incurs the well known drift

$$\mathrm{d}x^{\mathrm{FOR}}(t) = \sigma_x^{\mathrm{abs}}(t)\cdot\rho\cdot\sigma_Q(t)\mathrm{d}t - \kappa\cdot x^{\mathrm{FOR}}(t)\mathrm{d}t + \sigma_x^{\mathrm{abs}}(t)\mathrm{d}W_x(t) \quad (22)$$

$$x^{\mathrm{FOR}}(t) = \int_0^t \mathrm{e}^{-[K(t)-K(s)]}\cdot\sigma_x^{\mathrm{abs}}(s)\cdot[\rho\cdot\sigma_Q(s)\mathrm{d}s + \mathrm{d}W_x(s)] \quad (23)$$

which yields the ***Normal Quanto adjustment for the hazard process***:

$$\Xi(T) := \int_0^T\int_0^t \sigma_x^{\mathrm{abs}}(s)\cdot\rho\cdot\sigma_Q(s)\cdot\mathrm{e}^{-[K(t)-K(s)]}\,\mathrm{d}s\,\mathrm{d}t \quad (24)$$

with

$$\Lambda^{\mathrm{FOR}}(T) = \Lambda^{\mathrm{DOM}}(T) + \Xi(T)\,. \quad (25)$$

The Quanto adjustment for the normal hazard model translates directly into an equivalent deterministic hazard adjustment:

$$\bar{\Lambda}^{\mathrm{FOR}}(T) = \bar{\Lambda}^{\mathrm{DOM}}(T) + \Xi(T)\,. \quad (26)$$

For constant κ, σ_x^{abs}, and σ_Q, we obtain

$$\Xi(T) = \frac{\sigma_x^{\mathrm{abs}}\cdot\rho\cdot\sigma_Q}{\kappa^2}\cdot\left[\kappa T - 1 + \mathrm{e}^{-\kappa T}\right] \quad (27)$$

$$\approx \tfrac{1}{2}\sigma_x^{\mathrm{abs}}\cdot\rho\cdot\sigma_Q\cdot T^2\cdot\left[1 - \tfrac{1}{3}\kappa T + \tfrac{1}{12}\kappa^2T^2 + \mathcal{O}(\kappa^3T^3)\right] \quad (28)$$

for small κT.

Approximating the CDS spread for maturity T by the mean hazard via

$$s(T) \approx (1-R)\cdot\bar{\Lambda}(T)/T\ , \qquad (29)$$

we arrive at the following for the ***Normal Quanto CDS spread***

$$s^{\text{FOR}}(T) \approx s^{\text{DOM}}(T) + \frac{\sigma_X^{\text{abs}}\cdot\rho\cdot\sigma_Q}{\kappa}\cdot\left[1-\frac{\left(1-\mathrm{e}^{-\kappa T}\right)}{\kappa T}\right]\cdot(1-R)\ . \qquad (30)$$

Further assuming that σ_X^{abs} is given as a percentage σ_X of the mean hazard rate as approximated in (29), i.e.,

$$\sigma_X^{\text{abs}} \equiv \sigma_X\cdot\bar{\Lambda}(T)/T\ , \qquad (31)$$

we obtain

$$s^{\text{FOR}}(T) \approx s^{\text{DOM}}(T)\cdot[1+\eta(T)] \qquad (32)$$

with

$$\eta(T) = \underline{\underline{\sigma_X\cdot\rho\cdot\sigma_Q}}\cdot\left[1-\frac{\left(1-\mathrm{e}^{-\kappa T}\right)}{\kappa T}\right]\Big/\kappa\ . \qquad (33)$$

It seems that this model is capable of producing the very sizeable Quanto spreads we observe in the market.

Unfortunately, however, this has the side effect of allowing for the hazard process to become negative with significant probability, given by

$$p_-(T) = \Phi\left(\frac{\frac{1}{2}\mathrm{V}[\Lambda(T)]-\bar{\Lambda}(T)}{\sqrt{\mathrm{V}[\Lambda(T)]}}\right) \qquad (34)$$

with $\mathrm{V}[\Lambda(T)]$ as in (19) or (20).

We emphasize that the probability $p_-(T)$ is *not just the probability of the instantantaneous hazard rate going temporarily negative*, but the probability that the *cumulative hazard rate integral* becomes negative.

Such realizations are not desirable in any credit derivatives model since they represent a serious inversion of real world economics:

on such filtrations, the otherwise defaultable reference credit is likely to undergo a ***reverse default*** without ever having defaulted in the first place!

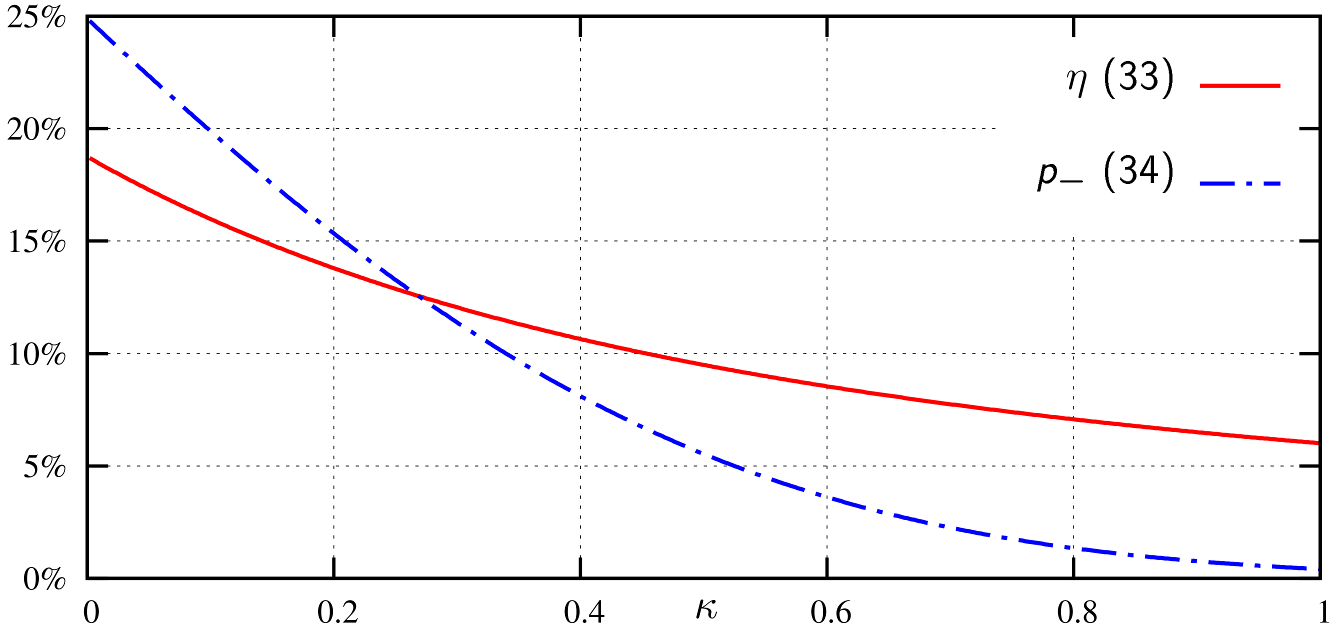

Figure 3: The relative Quanto spread adjustment coefficient η given in (33) and the associated probability p_- of $\Lambda(T)$ being negative at maturity T given in (34) for $\bar{\lambda} = 3\%$, $\sigma_x = 100\%$, $\sigma_Q = 15\%$, $\rho = 50\%$, and $T = 5$ as a function of mean reversion speed κ.

- The Quanto adjustment (27) scales like $\sim T^2$ for small T.
- This means that short to medium-dated Quanto CDS have invariably a very small FX exposure which is usually not in agreement with the Quanto CDS market, cf., e.g., the behaviour of the 2y versus the 1y point of the implied Quanto CDS (implied from USD - EUR quotes) in figure 1.
- This feature of the normal hazard model holds unless we take the limit $\kappa \to \infty$ whilst keeping $\alpha := \sigma_X^{\text{abs}}/\kappa$ constant, in which case (27) becomes $\Xi(T) = \alpha \cdot \rho \cdot \sigma_Q T$ and thus $\eta = \alpha \cdot \rho \cdot \sigma_Q$.

- Recall, however, that the Ornstein-Uhlenbeck process (13), in the limit of $\kappa \to \infty$ with $\sigma_x^{\text{abs}}/\sqrt{\kappa}$ kept constant (note the difference given by the square root) converges to *standard white noise*[2].

- This means that, in order to attain a linear scaling of the Quanto adjustment $\Xi(T)$ with maturity T, the hazard rate process has to take the form of a white noise process with infinite variance, which is (arguably) not a well defined concept.

- In addition, as we take the limit $\kappa \to \infty$ whilst keeping $\alpha := \sigma_x^{\text{abs}}/\kappa$ constant, the probability of the *instantaneous hazard rate* being negative converges to $^1/_2$ which is equally unsavoury.

[2]Standard white noise is the Gaussian process that is standard-normally distributed for all time horizons and auto-independent, i.e., any realization at time t is independent from the process's realization at time t' if $t \neq t'$.

- In the limit $\kappa \to \infty$ whilst keeping $\alpha := \sigma_x^{\text{abs}}/\kappa$ constant, the *hazard process* itself converges to a process with independent increments.

- **It is the independence of the hazard process's increments that provides us with the maturity-independence of the Quanto CDS adjustment**

$$\eta = \alpha \cdot \rho \cdot \sigma_Q \ .$$

Avoiding reverse defaults

- The hazard process is chosen to be a process that is non-decreasing ***with independent increments***.
- Non-decreasing: this avoids *reverse defaults*.
- Independent increments: this gives an economically more suitable scaling of the hazard's variance with maturity, and thus a more realistic Quanto CDS adjustment.
- Specifically, the hazard process is set to be given by a Gamma process (with potentially time-varying intensity ν):

$$\Lambda(T) = \int_0^T \nu(t) \cdot \mathrm{d}x(t)\ . \tag{35}$$

The time-homogeneous Gamma process $x(t)$ has the properties

$$x(t) \sim \psi\left(x(t); t/\sigma_x^2, 1/\sigma_x^2\right) \tag{36}$$

$$\psi(x; \alpha, \beta) = x^{\alpha-1}\beta^{\alpha}\mathrm{e}^{-\beta x} \Big/ \Gamma(\alpha) \tag{37}$$

$$\mathrm{E}\left[x(t)\right] = t \tag{38}$$

$$\mathrm{V}\left[x(t)\right] = \sigma_x^2 \cdot t \tag{39}$$

and the Laplace transform

$$\mathrm{E}\left[\mathrm{e}^{-\omega\cdot x(t)}\right] = \left(1 + \omega \cdot \sigma_x^2\right)^{-t/\sigma_x^2}\ . \tag{40}$$

From

$$\mathrm{e}^{-\bar{\Lambda}(T)} := p_{\text{Non-default}}(T) = \mathrm{E}\left[\mathrm{e}^{-\int_0^T \nu(t)\cdot \mathrm{d}x(t)}\right] \tag{41}$$

we can derive

$$\nu(t) = \left(\mathrm{e}^{\dot{\bar{\Lambda}}(t)\cdot\sigma_x^2} - 1\right) \Big/ \sigma_x^2\ . \tag{42}$$

For the foreign exchange rate, we keep a log-normal distribution as given in equation (10), i.e.,

$$Q(T) = \hat{Q}(T) \cdot \mathrm{e}^{-\frac{1}{2}\hat{\sigma}_Q^2 T + \hat{\sigma}_Q \cdot Z_Q} \tag{43}$$

for a Gaussian process Z_Q with variance

$$\mathrm{V}\left[Z_Q(t)\right] = t \; . \tag{44}$$

What about the codependence of $x(t)$ and $Z_Q(t)$?

A mathematically appealing approach is to start with three independent but otherwise identical Gamma processes

$$\tilde{x}_Q(t), \quad x_{\text{common}}(t), \quad \text{and} \quad \tilde{x}(t)$$

that all have the same statistical properties as $x(t)$.

We now compose two *dependent* Gamma processes $x(t)$ and $x_Q(t)$.

Over each time step $\mathrm{d}t$,

$$\text{with probability } \rho \quad \Longrightarrow \quad \mathrm{d}x(t) = \mathrm{d}x_Q(t) = \mathrm{d}x_{\text{common}}(t)$$

$$\text{else} \quad \Longrightarrow \quad \begin{cases} \mathrm{d}x(t) &= \mathrm{d}\tilde{x}(t) \\ \mathrm{d}x_Q(t) &= \mathrm{d}\tilde{x}_Q(t) \; . \end{cases}$$

The two Gamma processes $x(t)$ and $x_Q(t)$ constructed in this fashion are connected via a *Gamma* copula with correlation ρ.

In order to translate $x_Q(t)$ into a Gaussian process, we quantile-map it to a normal distribution according to

$$Z_Q(T) := \Phi^{-1}\left(\Gamma(x_Q(T); T/\sigma_x^2, 1/\sigma_x^2)\right) \cdot \sqrt{T}\,. \tag{45}$$

- Z_Q is a Gaussian process.
- Z_Q is not a diffusion.
- Z_Q is not even a Lévy process.
- One could argue about the conditional drift of Z_Q.
- Since the aim of the modelling within this scope is always limited to calculations that decompose into linear sums over expectations for different time horizons:

  no path-dependence concerns arise from this approximation for the joint T-maturity distribution of $Z_Q(T)$ and $x(T)$.

- To incorporate an FX smile, one could add a quantile map layer $Z_Q \to \chi(Z_Q)$.

What does this look like?

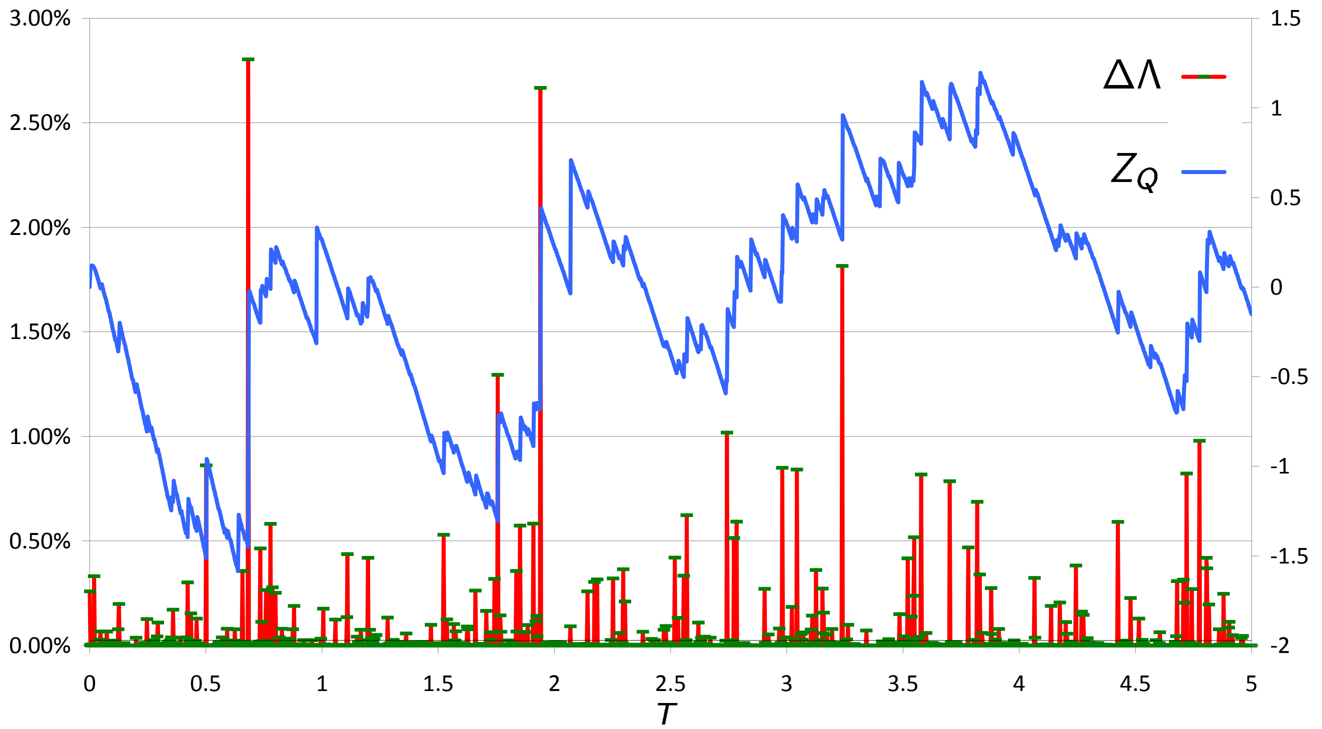

Figure 4: Sample paths of the Gaussian FX driver and Gamma hazard increments ($\Delta t = 1$day) under the Gamma copula for $\rho = 75\%$, $\sigma_x = 25\%$, and $\bar{\lambda} = 10\%$.

We note that a drawback to this setting is that, numerically, we end up having to compute multiple nested numerical integrations.

Since we are at this stage at liberty to choose a codependence structure not just for the sake of mathematical elegance, but alternatively for the sake of numerical tractability, we prefer to consider an approximation to a fully fledged instantaneous process correlation specification.

As it turns out, for practical purposes, a reasonable approximation for the Gamma copula connecting $Z_Q(T)$ and $x(T)$ is to use a Gaussian copula.

We shall later demonstrate numerically the quality of the approximation.

The Quanto non-default probability for the Gamma hazard model is

$$
\begin{aligned}
p^{\mathrm{FOR}}(T) &= \mathrm{e}^{-\bar{\Lambda}^{\mathrm{FOR}}(T)} \\
&= \mathsf{E}^{\mathcal{M}\left(P_T^{\mathrm{FOR}}\right)}\left[\mathrm{e}^{-\Lambda(T)}\right] \\
&= \frac{\mathsf{E}^{\mathcal{M}\left(P_T^{\mathrm{DOM}}\right)}\left[\mathrm{e}^{-\Lambda(T)}\cdot Q(T)\right]}{\mathsf{E}^{\mathcal{M}\left(P_T^{\mathrm{DOM}}\right)}\left[Q(T)\right]} && (46) \\
&= \mathsf{E}\left[\mathrm{e}^{-\frac{1}{2}\hat{\sigma}_Q^2 T+\hat{\sigma}_Q Z_Q(T)-\int_0^T \nu(t)\cdot \mathrm{d}x(t)}\right]\,. && (47)
\end{aligned}
$$

For constant parameters with

$$\nu = \left(\mathrm{e}^{\bar\lambda\cdot\sigma_x^2} - 1\right) \Big/ \sigma_x^2 \,, \tag{48}$$

this equals

$$\mathrm{e}^{-\bar\Lambda^{\mathrm{FOR}}(T)} = \mathsf{E}\left[\mathrm{e}^{-\frac{1}{2}\hat\sigma_Q^2\rho^2 T + \hat\sigma_Q\rho\sqrt{T}\cdot\Phi^{-1}\left(\Gamma(x;T/\sigma_x^2,1/\sigma_x^2)\right)-\nu\cdot x}\right] \tag{49}$$

$$= \mathsf{E}\left[\mathrm{e}^{-\frac{1}{2}\hat\sigma_Q^2\rho^2 T + \hat\sigma_Q\rho\sqrt{T}\cdot z - \nu\cdot\Gamma^{-1}\left(\Phi(z);T/\sigma_x^2,1/\sigma_x^2\right)}\right] \tag{50}$$

$$= \mathsf{E}\left[\mathrm{e}^{-\nu\cdot\Gamma^{-1}\left(\Phi(z+\hat\sigma_Q\rho\sqrt{T});T/\sigma_x^2,1/\sigma_x^2\right)}\right] \tag{51}$$

with z being an independent standard normal variate.

Following the spirit of Watson's lemma [Olv74, oST] and the idea of saddle-point approximations [Dan54], we can derive an *asymptotic expansion for large T* and constant parameters:

- The log-Gamma variate $\mathrm{e}^{-\nu\cdot x}$ converges in distribution to a log-normal variate with expectation $\mathrm{e}^{-\bar\lambda T}$, i.e.,

$$\mathrm{e}^{-\nu x} \approx \mathrm{e}^{-\bar\lambda T} \cdot \mathrm{e}^{-\frac{1}{2}\mathsf{V}[\nu x] - \sqrt{\mathsf{V}[\nu x]}\cdot z_x} \,. \tag{52}$$

with

$$\mathsf{V}[\nu x] = \left(\mathrm{e}^{\bar\lambda\sigma_x^2} - 1\right)^2 \cdot \frac{T}{\sigma_x^2} \tag{53}$$

$$\approx \bar\lambda^2\sigma_x^2 T \quad \text{for small } \bar\lambda. \tag{54}$$

- Hence, with $Z_Q(T) \sim \mathcal{N}(0, T)$

$$\mathsf{E}\left[\mathrm{e}^{-\nu x}\mathrm{e}^{-\frac{1}{2}\sigma_Q^2 T + \sigma_Q Z_Q(T)}\right] \approx \mathrm{e}^{-\bar\lambda T\cdot\left(1+\sigma_Q\rho\frac{\sqrt{\mathsf{V}[\nu x]}}{\bar\lambda\sqrt{T}}\right)} \tag{55}$$

- Thus we have for the spread,

$$s^{\text{FOR}}(T) \approx s^{\text{DOM}}(T) \cdot [1+\eta] \tag{56}$$

with the *relative Quanto spread adjustment*

$$\eta \approx \sigma_Q \cdot \rho \cdot \frac{e^{\bar\lambda\sigma_x^2}-1}{\bar\lambda\sigma_x} \tag{57}$$
$$\approx \sigma_Q \cdot \rho \cdot \sigma_x \tag{58}$$

for small $\bar\lambda$.

- Note that this has the same (desirable) form as the limit $T \to \infty$ of the normal hazard model's adjustment (33).

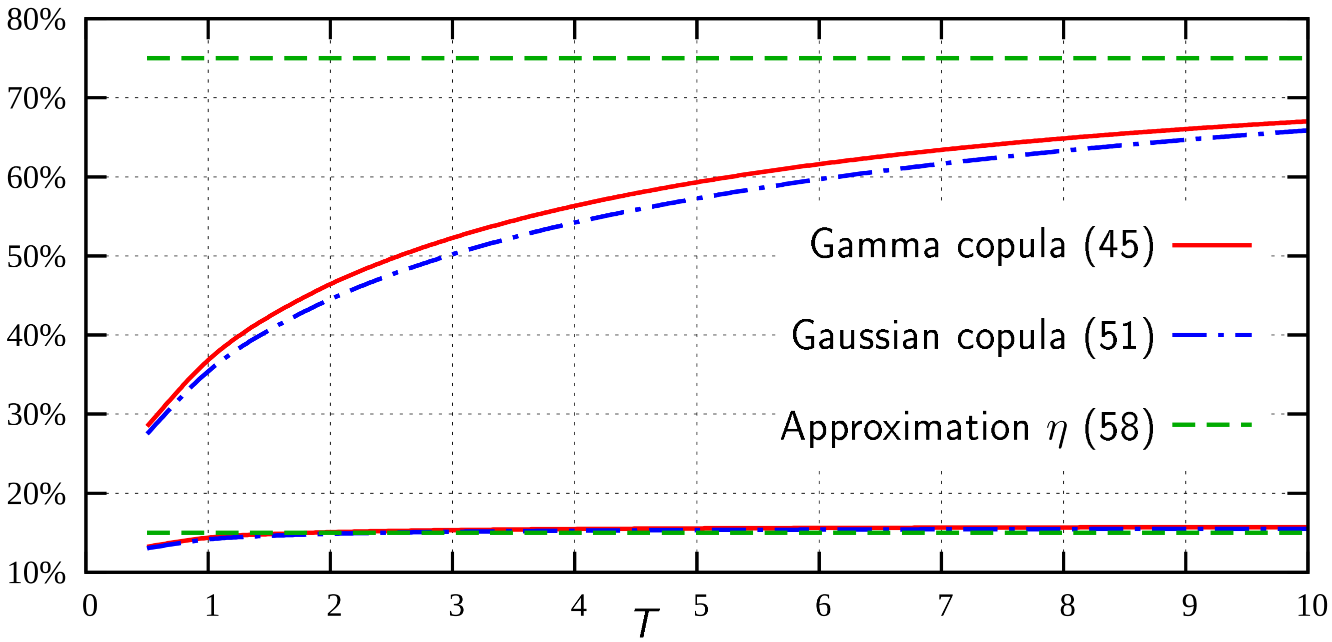

Figure 5: Log-relative non-default probability quanto difference $\frac{\ln p^{\text{FOR}}(T)}{\ln p^{\text{DOM}}(T)} - 1$ for $\hat\sigma_Q = 20\%$, $\rho = 75\%$, $\bar\lambda = 3\%$, and $\sigma_x = 100\%$ (lower three curves) and $\sigma_x = 500\%$ (upper three curves). The two copula calculations were done with 1048575 Sobol' draws.

- The original Gamma copula formulation of the model resulting in equation (45) for the Gaussian log-FX process driver does not allow for negative correlation between the foreign exchange rate and the hazard process.
- This is in contrast to the approximate use of a Gaussian copula for codependence.
- We resolve this apparent deficiency by demanding that *correlation must always be positive in the Gamma hazard model*.
- When observed correlation is negative, this simply means that, within the Gamma hazard model, what is being observed, are, as far as the model is concerned, Quanto non-default probabilities.
- Actual Quantoing to the target currency is then implicitly, within the model, the inverse problem of undoing the model's Quanto effect.

Example:

- The sovereign CDS quote on the kingdom of Belgium in USD is associated with a market expectation that, upon default, the FX rate EURUSD would decrease.
- In this case, we imply EUR denominated CDS quotes on Belgium by solving the *inverse problem*, i.e., we effectively demand that the "domestic" CDS quote is in EUR, and is unknown, whereas the "Quanto" CDS quote is the one observed more liquidly in USD in the market.
- A standard root-finding procedure can then be used to solve for the "domestic" CDS rate from the basic model's analytics, which are now again based on an effectively positive correlation number since they relate to the inverse of the FX rate.

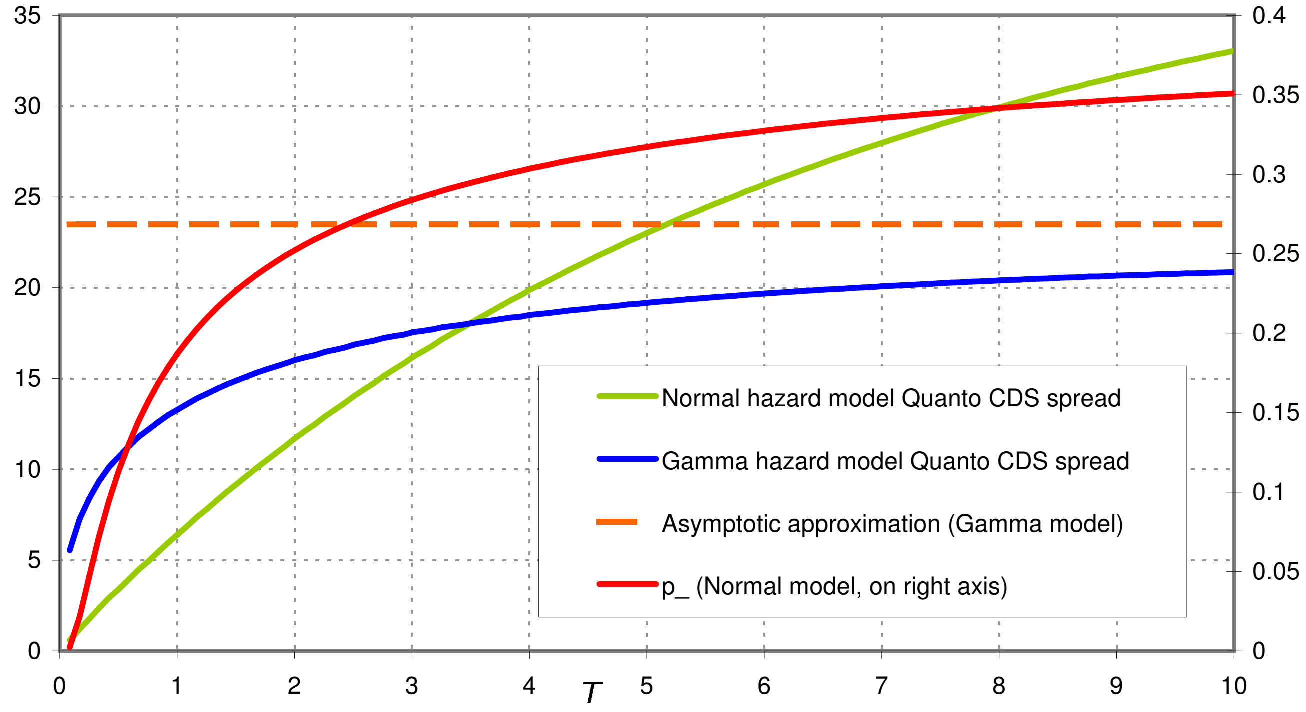

Figure 6: Quanto CDS spreads (foreign-domestic) with flat domestic CDS at 110bp, 40% recovery rate, $\rho = 30\%$, $\sigma_Q = 20\%$. For the normal model: $\sigma_x^{\text{abs}} = 3.8685\%$ and $\kappa = 25\%$. For the Gamma model: $\sigma_x^\Gamma = 356\%$. The asymptotic approximation is given in (58), and p_- in (34).

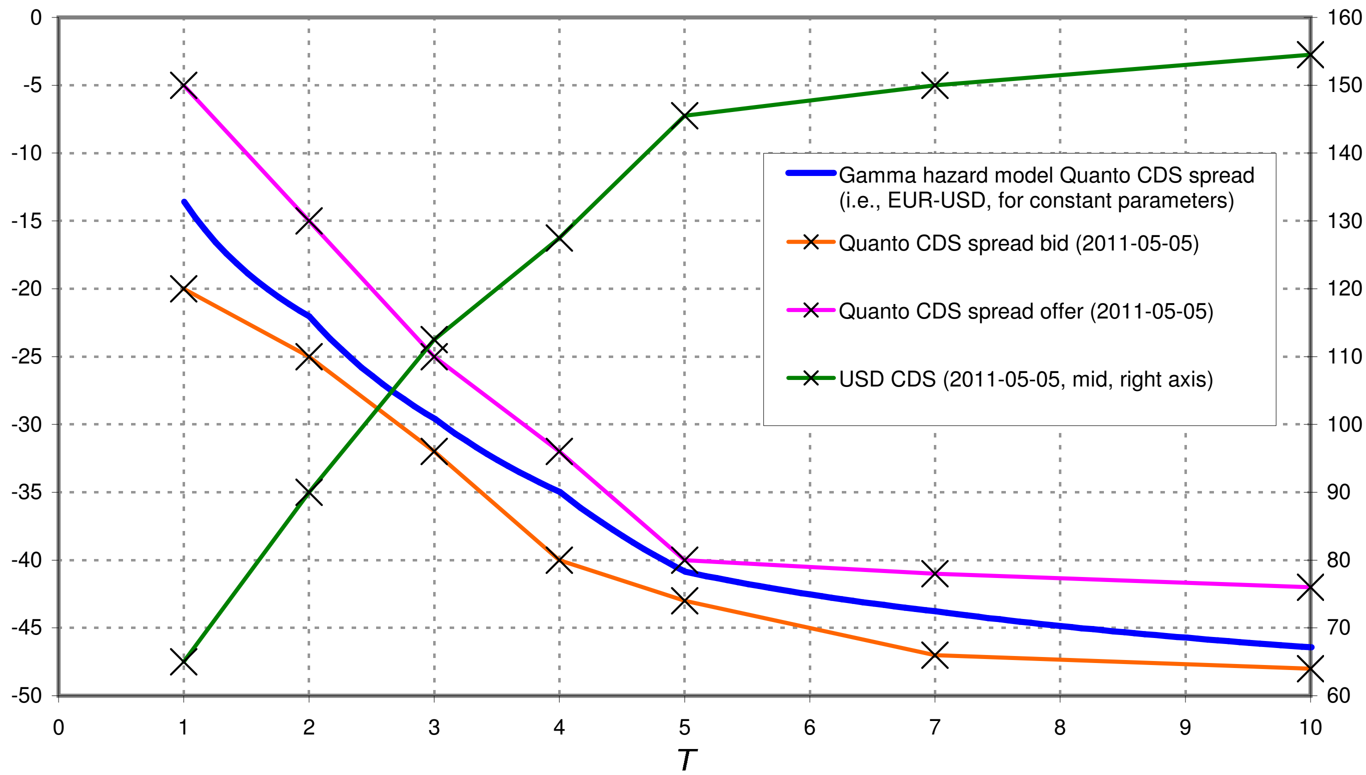

Figure 7: Gamma hazard Quanto CDS spreads (EUR-USD) for Italy on 2011-05-05, with 50% recovery rate, $\rho = -60\%$, $\sigma_Q = 20\%$ and $\sigma_x^\Gamma = 390\%$.

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