Chapter 1

Introduction

This chapter introduces the fundamental terminologies of Finite Difference Methods (FDM), providing the reader with the essential context. It covers key approximation techniques, including forward, backward, and central difference methods.

1.1 Introduction to FDM

Finite Difference Methods (FDM) are widely used numerical techniques for solving differential equations. By approximating derivatives using finite differences, these methods convert differential equations into systems of algebraic equations that can be solved numerically. Figure 1.1 illustrates an example of this technique applied to the function $f(x) = x^2$.

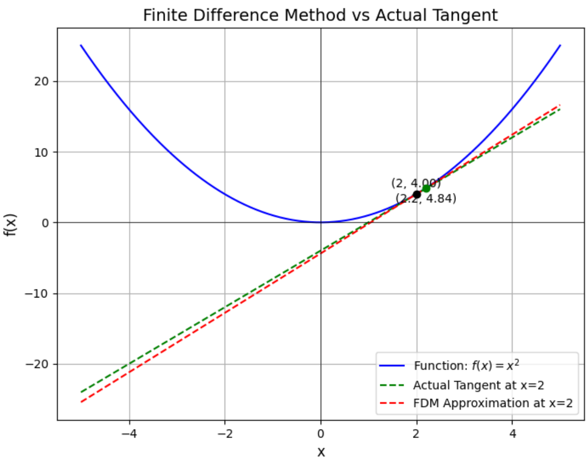

Figure 1.1: Finite difference approximation for $f(x) = x^2$

This section presents the numerical approximation techniques utilised for discretising continuous differential equations. Finite difference methods approximate derivatives through algebraic expressions, enabling the solution of partial differential equations.

1.1.1 Forward Difference Approximation

The forward difference approximation for the first derivative of a function $f(x)$ is derived by expanding $f(x + h)$ using Taylor's theorem around x.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f^{(3)}(x) + \dots$$

By rearranging terms and ignoring higher-order terms, we can approximate the first derivative as:

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

Thus, the forward difference approximation for the first derivative is:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad \text{(Forward Difference, 1st Order Accuracy)} \tag{1.1}$$

1.1.2 Backward Difference Approximation

The backward difference approximation is derived by expanding $f(x-h)$ using Taylor's expansion around x.

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f^{(3)}(x) + \dots$$

Rearranging terms and ignoring higher-order terms, the first derivative can be approximated as:

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h)$$

Thus, the backward difference approximation for the first derivative is:

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} \quad \text{(Backward Difference, 1st Order Accuracy)} \tag{1.2}$$

1.1.3 Central Difference Approximation

The central difference approximation is obtained by subtracting the Taylor expansion of $f(x-h)$ from that of $f(x+h)$.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f^{(3)}(x) + \dots$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f^{(3)}(x) + \dots$$

By subtracting these two expansions, we obtain:

$$f(x+h) - f(x-h) = 2hf'(x) + O(h^3)$$

Thus, the central difference approximation for the first derivative is:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \quad \text{(Central Difference, 2nd Order Accuracy)} \tag{1.3}$$

1.1.4 Second Derivative Approximation

To approximate the second derivative, we add the Taylor expansions of $f(x+h)$ and $f(x-h)$.

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f^{(3)}(x) + \dots \\
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f^{(3)}(x) + \dots
\end{aligned}
$$

Adding these expansions gives:

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4)$$

Rearranging the terms, we obtain the approximation for the second derivative:

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \quad \text{(Second Derivative, 2nd Order Accuracy)} \tag{1.4}$$

Chapter 2

The Black-Scholes-Merton Differential Equation

This chapter provides an in-depth exploration of the dynamics of the Black-Scholes-Merton (BSM) equation, along with the key assumptions that underlie its formulation. We will examine the differential equation in its partial differential form, where the option price is a function of the asset price and time, denoted as $V(S,t)$. Additionally, we will discuss the critical assumptions that lead to the derivation of this equation, which forms the foundation for much of modern option pricing theory.

2.1 The Black-Scholes-Merton Equation

The BSM equation is a parabolic partial differential equation (PDE) used to model the price dynamics of options over time. In its most common form, the equation relates the option price V(S,t) to the asset price S, time t, and other key parameters such as volatility σ, risk-free rate r, and dividend yield q. The equation is expressed as:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.1}$$

This equation forms the backbone of option pricing models to find V(S,t), the fair price of the option.

2.2 Assumptions Underlying the BSM Equation

The Black-Scholes-Merton equation is derived under several key assumptions, which are fundamental to its application in pricing financial derivatives. These assumptions include:

- **Lognormal Distribution of Asset Prices**: The price of the underlying asset follows a geometric Brownian motion, implying that asset prices are log-normally distributed.
- **No Arbitrage**: The model assumes that there are no arbitrage opportunities in the market, meaning it is impossible to make a risk-free profit.
- **Constant Volatility**: The volatility σ of the asset price is assumed to be constant over time, which simplifies the modeling of price dynamics.
- **Constant Risk-Free Rate**: The risk-free interest rate r is also assumed to remain constant over the life of the option.

- **Frictionless Markets**: The model assumes that there are no transaction costs, taxes, or other frictions in the market.
- **European-Style Options**: The equation is primarily designed for European options, which can only be exercised at expiration, rather than American options, which can be exercised at any time before expiration.
- **No Dividends (Optional)**: In the simplest form of the model, the underlying asset does not pay dividends. However, this can be adjusted for by including the dividend yield q in the equation.

These assumptions, while idealised, form the foundation of the Black-Scholes-Merton framework. In practice, some of these assumptions may be relaxed or adjusted to account for real-world complexities.

2.3 Schemes for differential equation

The following section focuses on discretising the BSM differential equation and explores various numerical schemes that approximate solutions to this equation, providing a foundation for practical option pricing algorithms. Before diving into the schemes, here are the discretisation methods and the boundary conditions for options:

1. **Discretisation:** The time and asset price domains are discretised into a grid. Let Δt be the time step and ΔS be the asset price step. Define:
   - $S_i = i\Delta S$ for $i = 0, 1, \ldots, N$
   - $t_n = n\Delta t$ for $n = 0, 1, \ldots, M$
2. **Boundary and Initial Conditions:**
   - **Call Options:**
     - **At Expiry:** $V(S, T) = \max(S - K, 0)$
     - **At Lower Boundary:** $V(0, t) = 0$, for $0 \leq t \leq T$
     - **At Upper Boundary:** $V(S_{\max}, t) \approx S_{\max} - K$
   - **Put Options:**
     - **At Expiry:** $V(S, T) = \max(K - S, 0)$
     - **At Lower Boundary:** $V(0, t) = K$, for $0 \leq t \leq T$
     - **At Upper Boundary:** $V(S_{\max}, t) = 0$

2.4 Von Neumann Stability Analysis on PDEs

Von Neumann stability analysis is a method used to analyse the stability of numerical schemes for solving partial differential equations (PDEs). This analysis primarily applies to linear PDEs and helps determine whether small perturbations in the initial conditions will grow or decay over time.

2.4.1 General Form of the PDE

Consider the following general form of a linear PDE:

$$\frac{\partial u}{\partial t} = \mathcal{L}(u)$$

where $u(x,t)$ is the solution we are solving for, and $\mathcal{L}(u)$ is a linear spatial differential operator acting on u. For example, $\mathcal{L}(u)$ could represent second-order spatial derivatives as seen in the heat equation.

2.4.2 Discretisation of the PDE

Let the grid points be defined as $u_j^n = u(x_j, t_n)$, where $x_j = j\Delta x$ and $t_n = n\Delta t$, with Δx and Δt representing the spatial and time step sizes, respectively.

A common discretization is:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \mathcal{L}_h(u_j^n)$$

where $\mathcal{L}_h(u_j^n)$ represents the discrete approximation of the spatial operator $\mathcal{L}(u)$.

2.4.3 Von-Neumann Stability Analysis

The Von Neumann analysis is based on the assumption that the error in the numerical solution can be represented as a Fourier series. Consider the error at a grid point u_j^n to be written in terms of Fourier modes:

$$\epsilon^n = \xi^n e^{ik_m x}$$

where ξ^n represents the amplification factor, k_m is the wave number, and $i = \sqrt{-1}$. The goal of the stability analysis is to determine conditions under which the amplification factor $|\xi| \leq 1$ for all modes.

2.5 Explicit Scheme

The explicit scheme, also known as the forward Euler method, is one of the simplest approaches to discretise the BSM differential equation. This method approximates the option price at future time steps based on known values at the current time step. While straightforward to implement, the explicit scheme requires careful consideration of stability conditions to ensure accurate results. To implement the explicit finite difference scheme, the following steps are typically followed:

1. **Finite Difference Approximations:**
   - The time derivative is approximated using a forward difference:
   $$\frac{\partial V}{\partial t} \approx \frac{V_i^{n+1} - V_i^n}{\Delta t}$$
   - The first spatial derivative is approximated using a central difference:
   $$\frac{\partial V}{\partial S} \approx \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta S}$$
   - The second spatial derivative is approximated using a central difference:
   $$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{(\Delta S)^2}$$
2. **Updating the Values:** The explicit scheme updates the option price at each grid point using the formula derived from substituting the finite difference approximations into the Black-Scholes equation. The update formula is:

$$V_i^{n+1} = V_i^n + \Delta t \left(\frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{(\Delta S)^2} + rS_i \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta S} - rV_i^n \right) \tag{2.2}$$

2.5.1 Von-Neumann analysis for explicit scheme

Consider equation 2.2, the error term for V_i^{n+1} can be defined as:

$$\epsilon_i^n = V_i^n - \mathcal{V}_i^n$$

Here, $\mathcal{V}_i^n$ represents the approximated value of V_i^n. Since the approximated solution $\mathcal{V}_i^n$ must satisfy the discretised equation, the error term ϵ_i^n is also required to satisfy the same discretised equation. Therefore, equation 2.2 can be represented as:

$$\epsilon_i^{n+1} = \epsilon_i^n + \Delta t \left(\frac{1}{2}\sigma^2 S_i^2 \frac{\epsilon_{i+1}^n - 2\epsilon_i^n + \epsilon_{i-1}^n}{(\Delta S)^2} + rS_i \frac{\epsilon_{i+1}^n - \epsilon_{i-1}^n}{2\Delta S} - r\epsilon_i^n \right) \tag{2.3}$$

The equations 2.2 and 2.3 show the same growth or decay behaviour with respect to time. For linear differential equations with boundary conditions, the spatial variation of the error can be expanded in a finite Fourier series about S. Let us assume that the error can be expressed as a Fourier mode of the form:

$$\epsilon_i^n = \xi^n e^{ik_m S}$$

where:

- ϵ_i^n represents the error at the spatial grid point i and time step n
- ξ^n is the amplification factor, which determines how the error grows or decays with each time step
- k_m is the wave number
- S represents the spatial variable

The assumption permits substituting the Fourier mode into equation 2.3. Substituting the Fourier mode for the error into the discretised equation, we obtain:

$$\xi^{n+1} e^{ik_m S} = \xi^n e^{ik_m S} + \Delta t \Bigg(\frac{1}{2}\sigma^2 S_i^2 \frac{\xi^n e^{ik_m(S+\Delta S)} - 2\xi^n e^{ik_m S} + \xi^n e^{ik_m(S-\Delta S)}}{(\Delta S)^2} \\ + rS_i \frac{\xi^n e^{ik_m(S+\Delta S)} - \xi^n e^{ik_m(S-\Delta S)}}{2\Delta S} - r\xi^n e^{ik_m S} \Bigg)$$

Dividing the above equation by $\xi^n e^{ik_m S}$ yields:

$$\frac{\xi^{n+1}}{\xi^n} = 1 + \Delta t \left(\frac{1}{2}\sigma^2 S_i^2 \frac{e^{ik_m \Delta S} - 2 + e^{-ik_m \Delta S}}{(\Delta S)^2} + rS_i \frac{e^{ik_m \Delta S} - e^{-ik_m \Delta S}}{2\Delta S} - r \right)$$

Let $\theta = k_m \Delta S$, therefore, by Euler's identity:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \quad e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

Substituting these into the equation, we get:

$$e^{ik_m \Delta S} - 2 + e^{-ik_m \Delta S} = 2\cos(\theta) - 2$$

Thus, the equation becomes:

$$\frac{\xi^{n+1}}{\xi^n} = 1 + \Delta t \left(\sigma^2 S_i^2 \frac{\cos(\theta) - 1}{(\Delta S)^2} + rS_i \frac{i\sin(\theta)}{\Delta S} - r \right) \tag{2.4}$$

Let $\xi = \frac{\xi^{n+1}}{\xi^n}$. For the numerical scheme to remain stable, it is required that $|\xi| \leq 1$.

Rewriting equation 2.4 in the form of $1 + A + iB$, we get

$$A = \Delta t \left(\frac{\sigma^2 S_i^2}{(\Delta S)^2} (\cos(\theta) - 1) - r \right)$$

$$B = \Delta t \left(\frac{r S_i}{\Delta S} \sin(\theta) \right)$$

For stability, we require $|\xi|^2 \leq 1$. Let's calculate this:

$$|\xi|^2 = (1 + A + iB)(1 + A - iB) = (1 + A)^2 + B^2$$

For stability:

$$(1 + A)^2 + B^2 \leq 1$$

Expanding this:

$$1 + 2A + A^2 + B^2 \leq 1$$
$$2A + A^2 + B^2 \leq 0$$

Substituting the expressions for A and B:

$$2\Delta t \left(\frac{\sigma^2 S_i^2}{(\Delta S)^2} (\cos(\theta) - 1) - r \right) + \left(\Delta t \left(\frac{\sigma^2 S_i^2}{(\Delta S)^2} (\cos(\theta) - 1) - r \right) \right)^2 + \left(\Delta t \left(\frac{r S_i}{\Delta S} \sin(\theta) \right) \right)^2 \leq 0$$

This inequality should hold for all values of θ in $[0, \pi]$. The most restrictive case is when $\theta = \pi$, which gives $\cos(\theta) = -1$ and $\sin(\theta) = 0$. Substituting these:

$$2\Delta t \left(\frac{\sigma^2 S_i^2}{(\Delta S)^2} (-2) - r \right) + \left(\Delta t \left(\frac{\sigma^2 S_i^2}{(\Delta S)^2} (-2) - r \right) \right)^2 \leq 0$$

Simplifying:

$$-4\Delta t \frac{\sigma^2 S_i^2}{(\Delta S)^2} - 2r\Delta t + \Delta t^2 \left(\frac{4\sigma^4 S_i^4}{(\Delta S)^4} + \frac{4r\sigma^2 S_i^2}{(\Delta S)^2} + r^2 \right) \leq 0$$

This is a quadratic inequality in Δt. For it to be satisfied, we need:

$$\Delta t \leq \frac{2}{\frac{\sigma^2 S_i^2}{(\Delta S)^2} + r} \tag{2.5}$$

Equation 2.5 provides an upper bound on the time step Δt in terms of the spatial step ΔS, the volatility σ, the risk-free rate r, and the stock price S_i. This condition is particularly restrictive for large values of S_i, which is why implicit or Crank-Nicolson schemes are often preferred for the Black-Scholes-Merton equation.

2.6 Implicit Scheme

The implicit scheme, widely used for solving time-dependent partial differential equations, is unconditionally stable and allows for larger time steps without sacrificing accuracy. By treating the future time step implicitly, the method requires solving a system of linear equations at each step, making it computationally intensive but highly robust.

1. **Finite Difference Approximations:**

- The time derivative is approximated using forward difference:

$$\frac{\partial V}{\partial t} \approx \frac{V_i^{n+1} - V_i^n}{\Delta t}$$

- The first spatial derivative is approximated using a central difference at the next time step:

$$\frac{\partial V}{\partial S} \approx \frac{V_{i+1}^{n+1} - V_{i-1}^{n+1}}{2\Delta S}$$

- The second spatial derivative is approximated using a central difference at the next time step:

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{(\Delta S)^2}$$

2. **Forming the System of Equations:** The implicit scheme leads to a system of linear equations. Substituting the finite difference approximations into the Black-Scholes equation yields:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = \frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{(\Delta S)^2} + rS_i \frac{V_{i+1}^{n+1} - V_{i-1}^{n+1}}{2\Delta S} - rV_i^{n+1}$$

3. **Solving the System:** Rearranging the equation leads to a tridiagonal system of the form:

$$a_i V_{i-1}^{n+1} + b_i V_i^{n+1} + c_i V_{i+1}^{n+1} = d_i$$

where:

$$\begin{aligned}
a_i &= -\frac{1}{2}\sigma^2 S_i^2 \frac{\Delta t}{(\Delta S)^2} + rS_i \frac{\Delta t}{2\Delta S} \\
b_i &= 1 + \sigma^2 S_i^2 \frac{\Delta t}{(\Delta S)^2} + r\Delta t \\
c_i &= -\frac{1}{2}\sigma^2 S_i^2 \frac{\Delta t}{(\Delta S)^2} - rS_i \frac{\Delta t}{2\Delta S} \\
d_i &= V_i^n
\end{aligned}$$

4. **Updating the Values:** Solve the tridiagonal system at each time step to obtain the option prices V_i^{n+1} for all i. This can be done efficiently using the Thomas algorithm or other tridiagonal matrix algorithms.

2.6.1 Von Neumann Stability Analysis for Implicit BSM Scheme

We start with the implicit discretisation of the BSM equation:

$$V_i^n - V_i^{n+1} + \Delta t \left(\frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{(\Delta S)^2} + rS_i \frac{V_{i+1}^{n+1} - V_{i-1}^{n+1}}{2\Delta S} - rV_i^{n+1} \right) = 0 \tag{2.6}$$

Define the error term as before:

$$\epsilon_i^n = V_i^n - \mathcal{V}_i^n \tag{2.7}$$

where $\mathcal{V}_i^n$ is the approximated value. Substituting the error term into the implicit discretisation:

$$\epsilon_i^n - \epsilon_i^{n+1} + \Delta t \left(\frac{1}{2}\sigma^2 S_i^2 \frac{\epsilon_{i+1}^{n+1} - 2\epsilon_i^{n+1} + \epsilon_{i-1}^{n+1}}{(\Delta S)^2} + rS_i \frac{\epsilon_{i+1}^{n+1} - \epsilon_{i-1}^{n+1}}{2\Delta S} - r\epsilon_i^{n+1} \right) = 0 \tag{2.8}$$

Assume the error can be expressed as a Fourier mode:

$$\epsilon_j^n = \xi^n e^{ik_m S} \tag{2.9}$$

Substituting this into the error equation:

$$\begin{aligned}\xi^n e^{ik_m S} = \xi^{n+1} e^{ik_m S} + \Delta t \Bigg(\frac{1}{2}\sigma^2 S_i^2 \frac{\xi^{n+1} e^{ik_m(S+\Delta S)} - 2\xi^{n+1} e^{ik_m S} + \xi^{n+1} e^{ik_m (S-\Delta S)}}{(\Delta S)^2} \\ + rS_i \frac{\xi^{n+1} e^{ik_m(S+\Delta S)} - \xi^{n+1} e^{ik_m(S-\Delta S)}}{2\Delta S} - r\xi^{n+1} e^{ik_m S} \Bigg)\end{aligned} \tag{2.10}$$

Divide by $\xi^n e^{ik_m S}$ and re-arranging the terms, we get:

$$1 - \xi + \Delta t \left(\frac{1}{2}\sigma^2 S_i^2 \frac{\xi e^{ik_m \Delta S} - 2\xi + \xi e^{-ik_m \Delta S}}{(\Delta S)^2} + rS_i \frac{\xi e^{ik_m \Delta S} - \xi e^{-ik_m \Delta S}}{2\Delta S} - r\xi \right) = 0 \tag{2.11}$$

where $\xi = \frac{\xi^{n+1}}{\xi^n}$

Let $\theta = k_m \Delta S$. Using Euler's identity:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{2.12}$$

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta) \tag{2.13}$$

Substituting and simplifying:

$$1 - \xi + \Delta t \left(\sigma^2 S_i^2 \xi \frac{\cos(\theta) - 1}{(\Delta S)^2} + rS_i \xi \frac{i\sin(\theta)}{\Delta S} - r\xi \right) = 0 \tag{2.14}$$

Rearranging to solve for ξ:

$$\xi = \frac{1}{1 + \Delta t \left(\sigma^2 S_i^2 \frac{1-\cos(\theta)}{(\Delta S)^2} - rS_i \frac{i\sin(\theta)}{\Delta S} + r \right)} \tag{2.15}$$

For stability, we require $|\xi| \leq 1$. This is always satisfied for the implicit scheme because:

$$|\xi| = \left| \frac{1}{1 + A + iB} \right| \leq 1 \tag{2.16}$$

where

$$A = \Delta t \left(\sigma^2 S_i^2 \frac{1 - \cos(\theta)}{(\Delta S)^2} + r \right) \tag{2.17}$$

$$B = -\Delta t \left(rS_i \frac{\sin(\theta)}{\Delta S} \right) \tag{2.18}$$

Both A and B are real, and A is always non-negative. Therefore, $|1 + A + iB| \geq 1$, ensuring $|\xi| \leq 1$.

The implicit scheme for the BSM equation is unconditionally stable, meaning there are no restrictions on the choice of Δt in terms of ΔS, σ, r, or S_i. However, it requires solving a system of equations at each time step, which can be computationally more intensive.

Chapter 3

Results

The results of the finite difference methods are illustrated through the following plots. These visualisations highlight the accuracy and behaviour of the numerical solutions compared to analytical benchmarks.

3.1 Option Prices

The following plots present the call and put options results, comparing numerical solutions with their analytical counterparts. These comparisons provide insights into the accuracy and effectiveness of the applied pricing models.

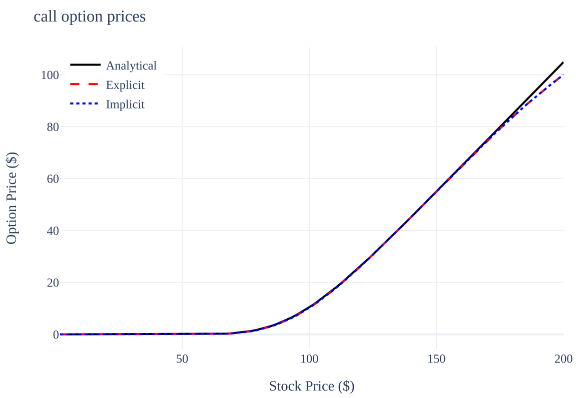

Figure 3.1: Call Option Prices

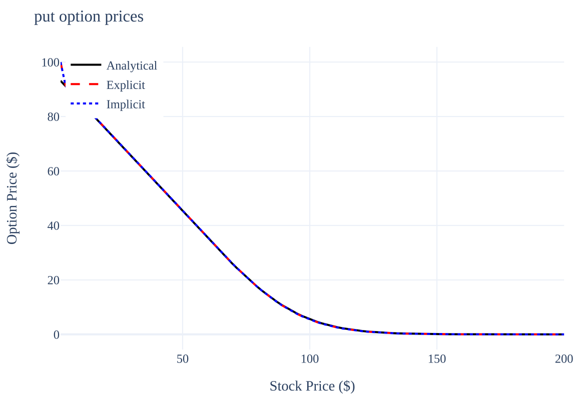

Figure 3.2: Put Option Prices

3.2 Surface Graphs

The surface graphs display the option prices computed using explicit and implicit finite difference methods for call options.

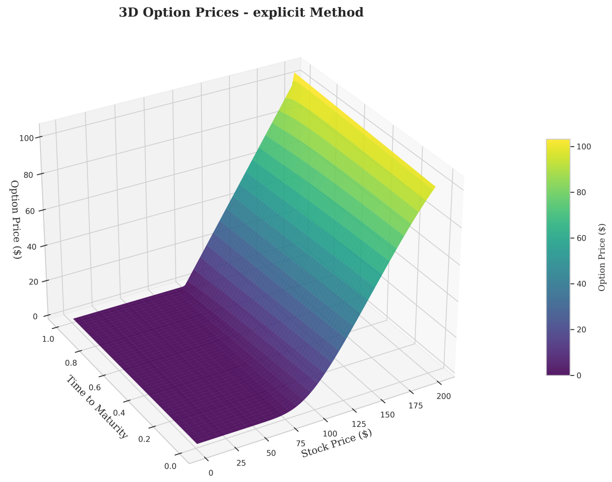

Figure 3.3: Explicit scheme surface graph

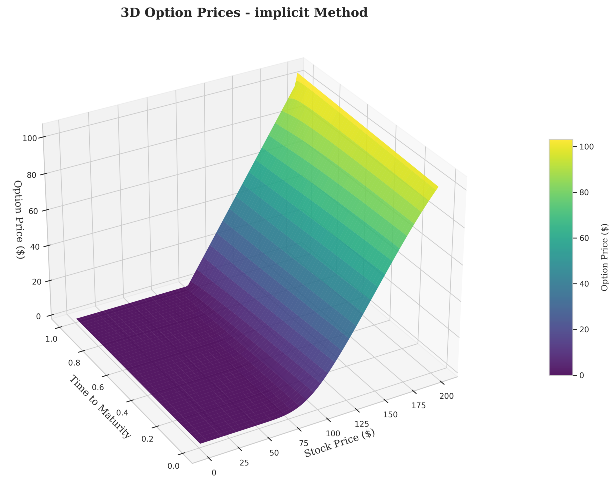

Figure 3.4: Implicit Scheme surface graph

3.3 Conclusion

In conclusion, the numerical methods discussed exhibit good accuracy in approximating option prices. However, while effective, they cannot fully match the precision of analytical solutions, highlighting the importance of understanding their strengths and limitations in financial modelling.