The Capital Asset Pricing Model: Some Empirical Tests

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ABSTRACT

Considerable attention has recently been given to general equilibrium models of the pricing of capital assets. Of these, perhaps the best known is the mean-variance formulation originally developed by Sharpe (1964) and Treynor (1961), and extended and clarified by Lintner (1965a; 1965b), Mossin (1966), Fama (1968a; 1968b), and Long (1972). In addition Treynor (1965), Sharpe (1966), and Jensen (1968; 1969) have developed portfolio evaluation models which are either based on this asset pricing model or bear a close relation to it. In the development of the asset pricing model it is assumed that (1) all investors are single period risk-averse utility of terminal wealth maximizers and can choose among portfolios solely on the basis of mean and variance, (2) there are no taxes or transactions costs, (3) all investors have homogeneous views regarding the parameters of the joint probability distribution of all security returns, and (4) all investors can borrow and lend at a given riskless rate of interest. The main result of the model is a statement of the relation between the expected risk premiums on individual assets and their "systematic risk." Our main purpose is to present some additional tests of this asset pricing model which avoid some of the problems of earlier studies and which, we believe, provide additional insights into the nature of the structure of security returns.

The evidence presented in Section II indicates the expected excess return on an asset is not strictly proportional to its β, and we believe that this evidence, coupled with that given in Section IV, is sufficiently strong to warrant rejection of the traditional form of the model given by (1). We then show in Section III how the cross-sectional tests are subject to measurement error bias, provide a solution to this problem through grouping procedures, and show how cross-sectional methods are relevant to testing the expanded two-factor form of the model. We show in Section IV that the mean of the beta factor has had a positive trend over the period 1931-65 and was on the order of 1.0 to 1.3% per month in the two sample intervals we examined in the period 1948-65. This seems to have been significantly different from the average risk-free rate and indeed is roughly the same size as the average market return of 1.3 and 1.2% per month over the two sample intervals in this period. This evidence seems to be sufficiently strong enough to warrant rejection of the traditional form of the model given by (1). In addition, the standard deviation of the beta factor over these two sample intervals was 2.0 and 2.2% per month, as compared with the standard deviation of the market factor of 3.6 and 3.8% per month. Thus the beta factor seems to be an important determinant of security returns.

Keywords: capital asset pricing, measurements, Cross-sectional Tests, Two-Factor Model, aggregation problem

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I. Introduction and Summary

Considerable attention has recently been given to general equilibrium models of the pricing of capital assets. Of these, perhaps the best known is the mean-variance formulation originally developed by Sharpe (1964) and Treynor (1961), and extended and clarified by Lintner (1965a; 1965b), Mossin (1966), Fama (1968a; 1968b), and Long (1972). In addition Treynor (1965), Sharpe (1966), and Jensen (1968; 1969) have developed portfolio evaluation models which are either based on this asset pricing model or bear a close relation to it. In the development of the asset pricing model it is assumed that (1) all investors are single period risk-averse utility of terminal wealth maximizers and can choose among portfolios solely on the basis of mean and variance, (2) there are no

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taxes or transactions costs, (3) all investors have homogeneous views regarding the parameters of the joint probability distribution of all security returns, and (4) all investors can borrow and lend at a given riskless rate of interest. The main result of the model is a statement of the relation between the expected risk premiums on individual assets and their "systematic risk." The relationship is

$$E\left(\tilde{R}_j\right) = E\left(\tilde{R}_M\right)\beta_j \tag{1}$$

where the tildes denote random variables and

$E\left(\tilde{R}_j\right) = \dfrac{E\left(\tilde{P}_t\right) - P_{t-1} + E\left(\tilde{D}_t\right)}{P_{t-1}} - r_{Ft} =$ expected excess returns on the *j*th asset

$\tilde{D}_t$ = dividends paid on the *j*th security at time t

r_{Ft} = the riskless rate of interest

$E\left(\tilde{R}_M\right)$ = expected excess returns on a "market portfolio" consisting of an investment in every asset outstanding in proportion to its value

$\beta_j = \dfrac{cov\left(\tilde{R}_j, \tilde{R}_M\right)}{\sigma^2\left(\tilde{R}_M\right)}$ = the "systematic" risk of the *j*th asset.

Relation 1 says that the expected excess return on any asset is directly proportional to its β. If we define α_j as

$$\alpha_j = E\left(\tilde{R}_j\right) - E\left(\tilde{R}_M\right)\beta_j$$

then (1) implies that the a on every asset is zero.

If empirically true, the relation given by (1) has wide-ranging implications for problems in capital budgeting, cost benefit analysis, portfolio selection, and for other economic problems requiring knowledge of the relation between risk and return. Evidence presented by Jensen (1968; 1969) on the relationship between the expected return and systematic risk of a large sample of mutual funds suggests that (1) might provide an adequate description of the relation between risk and return for securities. On the other hand, evidence presented by Douglas (1969), Lintner (1965a), and most recently Miller and Scholes (1972) seems to indicate the model does not provide a complete description of the structure of security returns. In particular, the work done by Miller and Scholes suggests that the α's on individual assets depend in a systematic way on their β's: that high-beta assets tend to have negative α's, and that low-beta stocks tend to have positive α's.

Our main purpose is to present some additional tests of this asset pricing model which avoid some of the problems of earlier studies and which, we believe, provide additional insights into the nature of the structure of security returns. All previous direct tests of the model have been conducted using cross-sectional methods; primarily regression of $\overline{R}_j$, the mean excess return over a time interval for a set of securities on estimates of the systematic risk, $\hat{\beta}_j$, of each of the securities. The equation

$$\overline{R}_j = \gamma_0 + \gamma_1 \hat{\beta}_j + \tilde{u}_j$$

was estimated, and contrary to the theory, γ_0 seemed to be significantly different from zero and γ_1 significantly different from $\overline{R}_M$, the slope predicted by the model. We shall show in Section III that, because of the structure of the process which appears to be generating the data, these cross-sectional tests of significance can be misleading and therefore do not provide direct tests of the validity of (1). In Section II we provide a more powerful time series test of the validity of the model, which is free of the difficulties associated with the cross-sectional tests. These results indicate that the usual form of the asset pricing model as given by (1) does not provide an accurate description of the structure of security returns. The tests indicate that the expected excess returns on high-beta assets are lower than (1) suggests and that the expected excess returns on low-beta assets are higher than (1) suggests. In other words, that high-beta stocks have negative α's and low-beta stocks have positive α's.

The data indicate that the expected return on a security can be represented by a two-factor model such as

$$E(\tilde{r}_j) = E(\tilde{r}_z)(1-\beta_j) + E(\tilde{r}_M)\beta_j \tag{2}$$

where the r's indicate total returns and $E(\tilde{r}_z)$ is the expected return on a second factor, which we shall call the "beta factor," since its coefficient is a function of the asset's β. After we had observed this phenomenon, Black (1970) was able to show that relaxing the assumption of the existence of riskless borrowing and lending opportunities provides an asset pricing model which implies that, in equilibrium, the expected return on an asset will be given by (2). His results furnish an explicit definition of the beta factor, $\tilde{r}_z$, as the return on a portfolio that has a zero covariance with the return on the market portfolio $\tilde{r}_M$. Although this model is entirely consistent with our empirical results (and provides a convenient interpretation of them), there are perhaps other plausible hypotheses consistent with the data (we shall briefly discuss several in Section V). We hasten to add that we have not attempted here to supply any direct tests of these alternative hypotheses.

The evidence presented in Section II indicates the expected excess return on an asset is not strictly proportional to its β, and we believe that this evidence, coupled with that given in Section IV, is sufficiently strong to warrant rejection of the traditional form of the model given by (1). We then show in Section III how the cross-sectional tests are subject to measurement error bias, provide a solution to this problem through grouping procedures, and show how cross-sectional methods are relevant to testing the expanded two-factor form of the model. Here we find that the evidence indicates the existence of a linear relation between risk and return and is therefore consistent with a form of the two-factor model which specifies the realized returns on each asset to be a linear function of the returns on the two factors $\tilde{r}_z$, and $\tilde{r}_M$,

$$\tilde{r}_j = \tilde{r}_z\left(1 - \beta_j\right) + \tilde{r}_M \beta_j + \tilde{w}_j \tag{2}$$

The fact that the α's of high-beta securities are negative and that the α's of low-beta securities are positive implies that the mean of the beta factor is greater than r_F. The traditional form of the capital asset pricing model as expressed by (1), could hold exactly, even if asset returns were generated by $(2')$, if the mean of the beta factor were equal to the risk-free rate. We show in Section IV that the mean of the beta factor has had a positive trend over the period 1931-65 and was on the order of 1.0 to 1.3% per month in the two sample intervals we examined in the period 1948-65. This seems to have been significantly different from the average risk-free rate and indeed is roughly the same size as the average market return of 1.3 and 1.2% per month over the two sample intervals in this period. This evidence seems to be sufficiently strong enough to warrant rejection of the traditional form of the model given by (1). In addition, the standard deviation of the beta factor over these two sample intervals was 2.0 and 2.2% per month, as compared with the standard deviation of the market factor of 3.6 and 3.8% per month. Thus the beta factor seems to be an important determinant of security returns.

II. Time Series Tests of the Model

A. Specification of the Model.

Although the model of (1) which we wish to test is stated in terms of expected returns, it is possible to use realized returns to test the theory. Let us represent the returns on any security by the "market model" originally proposed by Markowitz (1959) and extended by Sharpe (1963) and Fama (1968b)

$$\tilde{R}_j = E(\tilde{R}_j) + \beta_j \tilde{R}'_M + \tilde{e}_j \tag{3}$$

where $\tilde{R}'_M = \tilde{R}_M - E(\tilde{R}_M) =$ the "unexpected" excess market return, and $\tilde{R}'_M$ and $\tilde{e}_j$ are normally distributed random variables that satisfy:

$$E(\tilde{R}'_M) = 0 \tag{4a}$$

$$E(\tilde{e}_j) = 0 \tag{4b}$$

$$E(\tilde{e}_j \tilde{R}'_M) = 0 \tag{4c}$$

The specifications of the market model, extensively tested by Fama et al. (1969) and Blume (1968), are well satisfied by the data for a large number of securities on the New York Stock Exchange. The only assumption violated to any extent is the normality assumption[1]--the estimated residuals seem to conform to the infinite variance members of the stable class of distributions rather than the normal. There are those who would explain these discrepancies from normality by certain non-stationarities in the distributions (cf. Press (1967)), which still yield finite variances. However, Wise (1963)

[1] Note that (4c) can be valid even though R_M is a weighted average of the R_j and therefore R'_M contains e_j. This may be clarified as follows: taking the weighted sum of (3) using the weights, X_j, of each security in the market portfolio we know by the definition of R_M that $\sum_j X_j R_j = R_M \sum_j X_j \beta_j = 1$, and $\sum_j X_j e_j = 0$. Thus by the last equality we know $X_j e_j = -\sum_{i \neq j} X_i e_i$, and by substitution $E(e_j X_j e_j) = \left[e_j\left(-\sum_{i \neq j} X_i e_i\right)\right] = X_j \sigma^2(e_j)$, and this implies condition (4c) since $E(e_j R'_M) = X_j \sigma^2(e_j) + E\left[e_j \sum_{i \neq j} X_i e_i\right] = 0$.

has shown that the least-squares estimate of β_j in (3) is unbiased (although not efficient) even if the variance does not exist, and simulations by Blattberg and Sargent (1971) and Fama and Babiak (1967) also indicate that the least-squares procedures are not totally inappropriate in the presence of infinite variance stable distributions. For simplicity, therefore, we shall ignore the non-normality issues and continue to assume normally distributed random variables where relevant.[2] However, because of these problems caution should be exercised in making literal interpretations of any significance tests. Substituting from (1) for $E(\tilde{R}_j)$ in (3) we obtain

$$\tilde{R}_j = \tilde{R}_M \beta_j + \tilde{e}_j \tag{5}$$

where $\tilde{R}_M$ is the ex post excess return on the market portfolio over the holding period of interest. If assets are priced in the market such that (1) holds over each short time interval (say a month), then we can test the traditional form of the model by adding an intercept α_j to (5) and subscripting each of the variables by *t* to obtain

$$\tilde{R}_{jt} = \alpha_j + \beta_j \tilde{R}_{Mt} + \tilde{e}_{jt} \tag{6}$$

which, given the assumptions of the market model, is a regression equation. If the asset pricing and the market models given by (1), (3), and (4) are valid, then the intercept α_j in (6) will be zero. Thus a direct test of the model can be obtained by estimating (6) for a security over some time period and testing to see if α_j is significantly different from zero.[3,4]

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2 We could develop the model and tests under the assumption of infinite variance stable distributions, but this would unnecessarily complicate some of the analysis. We shall take explicit account of these distributional problems in some of the crucial tests of significance in Section IV.

3 Recall that the R_{jt} and R_{Mt}, are defined as excess returns. The model can be formulated with r_{Ft} omitted from (6) and therefore assumed constant (then $\alpha_j = r_F(1-\beta_j)$) or included as a variable (as we

B. An Aggregation Problem.

The test just proposed is simple but inefficient, since it makes use of information on only a single security whereas data is available on a large number of securities. We would like to design a test that allows us to aggregate the data on a large number of securities in an efficient manner. If the estimates of the α_j's were independent with normally distributed residuals, we could proceed along the lines outlined by Jensen (1968) and compare the frequency distributions of the "*t*" values for the intercepts with the theoretical distribution. However, the fact that the $\tilde{e}_{jt}$, are not cross-sectionally independent, (that is, $E(\tilde{e}_{jt}\tilde{e}_{it}) \neq 0$ for $i \neq j$, cf. King (1966); makes this procedure much more difficult.

One procedure for solving this problem which makes appropriate allowance for the effects of the non-independence of the residuals on the standard error of estimate of the average coefficient, $\tilde{\alpha}$, is to run the tests on grouped data. That is, we form portfolios (or groups) of the individual securities and estimate (6) defining $\tilde{R}_{Kt}$ to be the average return on all securities in the *K*th portfolio for time *t*. Given this definition of $\tilde{R}_{Kt}$, $\hat{\beta}_K$ will be the average risk of the securities in the portfolio and $\hat{\alpha}_K$ will be the

have done), which strictly requires them to be known for all *t*. But experiments with estimates obtained with the inclusion of r_{Ft} as a variable in (6) yield results virtually identical to those obtained with the assumption of constant r_F [and hence the exclusion of r_{Ft} as a variable in (6)], so we shall ignore this problem here. See also Roll (1969) and Miller and Scholes (1972) for a thorough discussion of the bias introduced through misspecification of the riskless rate. Miller and Scholes conclude as we do that these problems are not serious.

4 Unbiased measurement errors in $\hat{\beta}_j$ cause severe difficulties with the cross-sectional tests of the model, and it is important to note that the time series form of the tests given by (6) are free of this source of bias. Unbiased measurement errors in $\hat{\beta}_j$, which is estimated simultaneously with α_j in the time series formulation, cause errors in the estimate of α_j but no systematic bias. Measurement errors in R_{Mt}, may cause difficulties in both the cross-sectional and time series forms of the tests, but we

average intercept. Moreover, since the residual variance from this regression will incorporate the effects of any cross-sectional interdependencies in the $\tilde{e}_{jt}$ among the securities in each portfolio, the standard error of the intercept $\hat{\alpha}_K$ will appropriately incorporate the non-independence of $\tilde{e}_{jt}$.

In addition, we wish to group our securities such that we obtain the maximum possible dispersion of the risk coefficients, β_K. If we were to construct our portfolios by using the ranked values of the $\hat{\beta}_j$, we would introduce a selection bias into the procedure. This would occur because those securities entering the first or high-beta portfolio would tend to have positive measurement errors in their $\hat{\beta}_j$, and this would introduce positive bias in $\hat{\beta}_K$, the estimated portfolio risk coefficient. This positive bias in $\hat{\beta}_K$ will, of course, introduce a negative bias in our estimate of the intercept, $\hat{\alpha}_K$, for that portfolio. On the other hand, the opposite would occur for the lowest beta portfolio; its $\hat{\beta}_K$ would be negatively biased, and therefore our estimate of the intercept for this low-risk portfolio would be positively biased. Thus even if the traditional model were true, this selection bias would tend to cause the low-risk portfolios to exhibit positive intercepts and high-risk portfolios to exhibit negative intercepts. To avoid this bias, we need to use an instrumental variable that is highly correlated with $\hat{\beta}_j$, but that can be observed independently of $\hat{\beta}_j$. The instrumental variable we have chosen is simply an independent estimate of the β of the security obtained from past data. Thus when we estimate the group risk parameter on sample data not used in the ranking procedures, the measurement

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shall ignore this issue here. For an analysis of the problems associated with measurement errors in R_{Mt}, see Black and Jensen (1970), Miller and Scholes (1972), and Roll (1969).

errors in these estimates will be independent of the errors in the coefficients used in the ranking and we therefore obtain unbiased estimates of $\hat{\beta}_K$ and $\hat{\alpha}_K$.

C. The Data.

The data used in the tests to be described were taken from the University of Chicago Center for Research in Security Prices Monthly Price Relative File, which contains monthly price, dividend, and adjusted price and dividend information for all securities listed on the New York Stock Exchange in the period January, 1926-March, 1966. The monthly returns on the market portfolio R_{Mt} were defined as the returns that would have been earned on a portfolio consisting of an equal investment in every security listed on the NYSE at the beginning of each month. The risk-free rate was defined as the 30-day rate on U.S. Treasury Bills for the period 1948-66. For the period 1926-47 the dealer commercial paper rate[5] was used because Treasury Bill rates were not available.

D. The Grouping Procedure

<u>1. The ranking procedure.</u>

Ideally we would like to assign the individual securities to the various groups on the basis of the ranked β_j (the true coefficients), but of course these are unobservable. In addition we cannot assign them on the basis of the $\hat{\beta}_j$, since this would introduce the selection bias problems discussed previously. Therefore, we must use a ranking procedure that is independent of the measurement errors in the $\hat{\beta}_j$. One way to do this is to use part of the data--in our case five years of previous monthly data--to obtain estimates $\hat{\beta}_{j0}$, of the risk measures for each security. The ranked values of the $\hat{\beta}_{j0}$ are used to assign membership to the groups. We then use data from a subsequent time period to estimate the group risk coefficients $\hat{\beta}_K$, which then contain measurement errors for the individual securities, which are independent of the errors in $\hat{\beta}_{j0}$ and hence independent of the original ranking and independent among the securities in each group.

2. The stationarity assumptions.

The group assignment procedure just described will be satisfactory as long as the coefficients β_j are stationary through time. Evidence presented by Blume (1968) indicates this assumption is not totally inappropriate, but we have used a somewhat more complicated procedure for grouping the firms which allows for any non-stationarity in the coefficients through time.

We began by estimating the coefficient β_j, (call this estimate $\hat{\beta}_{j0}$) in (6) for the five-year period January, 1926-December, 1930 for all securities listed on the NYSE at the beginning of January 1931 for which at least 24 monthly returns were available. These securities were then ranked from high to low on the basis of the estimates $\hat{\beta}_{j0}$, and were assigned to ten portfolios[6]--the 10% with the largest $\hat{\beta}_{j0}$ to the first portfolio, and so on. The return in each of the next 12 months for each of the ten portfolios was calculated. Then the entire process was repeated for all securities listed as of January 1932 (for

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5 Treasury Bill rates were obtained from the Salomon Brothers & Hutzler quote sheets at the end of the previous month for the following month. Dealer commercial paper rates were obtained from Banking and Monetary Statistics, Board of Governors of the Federal Reserve System, Washington, D.C.

6 The choice of the number of portfolios is somewhat arbitrary. As we shall see below, we wanted enough portfolios to provide a continuum of observations across the risk spectrum to enable us to estimate the suspected relation between α_K and β_K.

which at least 24 months of previous monthly returns were available) using the immediately preceding five years of data (if available) to estimate new coefficients to be used for ranking and assignment to the ten portfolios. The monthly portfolio returns were again calculated for the next year. This process was then repeated for January 1933, January 1934, and so on, through January 1965.

TABLE 1
Total Number of Securities
Entering All Portfolios, by Year

Year	*Number of Securities*	*Year*	*Number of Securities*
1931	582	1949	893
1932	673	1950	928
1933	688	1951	943
1934	683	1952	966
1935	676	1953	994
1936	674	1954	1000
1937	666	1955	1006
1938	690	1956	994
1939	718	1957	994
1940	743	1958	1000
1941	741	1959	995
1942	757	1960	1021
1943	772	1961	1014
1944	778	1962	1024
1945	773	1963	1056
1946	791	1964	1081
1947	812	1965	1094
1948	842		

In this way we obtained 35 years of monthly returns on ten portfolios from the 1,952 securities in the data file. Since at each stage we used all listed securities for which at least 24 months of data were available in the immediately preceding five-year period, the total number of securities used in the analysis varied through time ranging from 582 to 1,094, and thus the number of securities contained in each portfolio changed from year to year.[7] The total number of securities from which the portfolios were formed at the beginning of each year is given in Table 1. Each of the portfolios may be thought of as a mutual fund portfolio, which has an identity of its own, even though the stocks it contains change over time.

E. The Empirical Results

1. The entire period.

Given the 35 years of monthly returns on each of the ten portfolios calculated as explained previously, we then calculated the least-squares estimates of the parameters α_K and β_K in (6) for each of the ten portfolios (K = 1, . . ., 10) using all 35 years of monthly data (420 observations). The results are summarized in Table 2. Portfolio number 1 contains the highest-risk securities and portfolio number 10 contains the lowest-risk securities. The estimated risk coefficients range from 1.561 for portfolio 1 to 0.499 for portfolio 10. The critical intercepts, the $\hat{\alpha}_K$, are given in the second line of Table 2 and the Student "*t*" values are given directly below them. The correlation between the portfolio returns and the market returns, $r(\tilde{R}_K, \tilde{R}_M)$, and the autocorrelation of the residuals, $r(\tilde{e}_t, \tilde{e}_{t-1})$, are also given in Table 2. The autocorrelation appears to be quite small and the correlation between the portfolio and market returns are, as expected, quite high. The standard deviation of the residuals $\sigma(\tilde{e}_K)$, the average monthly excess return $\overline{R}_K$, and the standard deviation of the monthly excess return, σ, are also given for each of the portfolios.

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7 Note that in order for the risk parameters of the groups β_K, to be stationary through time, our procedures require that firms leave and enter the sample symmetrically across the entire risk spectrum.

Note first that the intercepts $\hat{\alpha}$ are consistently negative for the high-risk portfolios $(\hat{\beta} > 1)$ and consistently positive for the low-risk portfolios $(\hat{\beta} < 1)$. Thus the high-risk securities earned less on average over this 35-year period than the amount predicted by the traditional form of the asset pricing model. At the same time, the low-risk securities earned more than the amount predicted by the model.

Table 2
Summary of Statistics for Time Series Tests, Entire Period (January, 1931-December, 1965)
(Sample Size for Each Regression =420)

Item*	*Portfolio Number* 1	2	3	4	5	6	7	8	9	10	$\bar{R}_M$
$\hat{\beta}$	1.5614	1.3838	1.2483	1.1625	1.0572	0.9229	0.8531	0.7534	0.6291	0.4992	1.0000
$\hat{\alpha} \bullet 10^2$	-0.0829	-0.1938	-0.0649	-0.0167	-0.0543	0.0593	0.0462	0.0812	0.1968	0.2012	
$t(\hat{\alpha})$	-0.4274	-1.9935	-0.7597	-0.2468	-0.8869	0.7878	0.7050	1.1837	2.3126	1.8684	
$r(\tilde{R}, \tilde{R}_M)$	0.9625	0.9875	0.9882	0.9914	0.9915	0.9833	0.9851	0.9793	0.9560	0.8981	
$r(\tilde{e}_t, \tilde{e}_{t-1})$	0.0549	-0.0638	0.0366	0.0073	-0.0708	-0.1248	0.1294	0.1041	0.0444	0.0992	
$\sigma(\tilde{e})$	0.0393	0.0197	0.0173	0.0137	0.0124	0.0152	0.0133	0.0139	0.0172	0.0218	
$\bar{R}$	0.0213	0.0177	0.0171	0.0163	0.0145	0.0137	0.0126	0.0115	0.0109	0.0091	0.0142
σ	0.1445	0.1248	0.1126	0.1045	0.0950	0.0836	0.0772	0.0685	0.0586	0.0495	0.0891

* $\bar{R}_M$ = average monthly excess returns, σ = standard deviation of the monthly excess returns, r = correlation coefficient.

The significance tests given by the "*t*" values in Table 2 are somewhat inconclusive, since only 3 of the 10 coefficients have "*t*" values greater than 1.85 and, as we pointed out earlier, we should use some caution in interpreting these "*t*" values since the normality assumptions can be questioned. We shall see, however, that due to the existence of some nonstationarity in the relations and to the lack of more complete aggregation, these results vastly understate the significance of the departures from the traditional model.

2. The subperiods.

In order to test the stationarity of the empirical relations, we divided the 35-year interval into four equal subperiods each containing 105 months. Table 3 presents a summary of the regression statistics of (6) calculated using the data for each of these periods for each of the ten portfolios. Note that the data for $\hat{\beta}$ in Table 3 indicate that, except for portfolios 1 and 10, the risk coefficients $\hat{\beta}_K$ were fairly stationary.

Note, however, in the sections for α and $t(\hat{\alpha})$ that the critical intercepts $\hat{\alpha}_K$, were most definitely non-stationary throughout this period. The positive α's for the high-risk portfolios in the first sub period (January, 1931-September, 1939) indicate that these securities earned more than the amount predicted by the model, and the negative α's for the low-risk portfolios indicate they earned less than what the model predicted. In the three succeeding subperiods (October, 1939-June, 1948; July, 1948-March, 1957, and April, 1957-December, 1965) this pattern was reversed and the departures from the model seemed to become progressively larger; so much larger that six of the ten coefficients in the last subperiod seem significant. (Note that all six coefficients are those with β's most different from unity--a point we shall return to. Thus it seems unlikely that these changes were the result of chance; they most probably reflect changes in the $\hat{\alpha}_K$'s).

Note that the correlation coefficients between $\tilde{R}_{Kt}$ and $\tilde{R}_{Mt}$ given in Table 2 for each of the portfolios are all greater than 0.95 except for portfolio number 10. The lowest of the 40 coefficients in the subperiods (not shown) was 0.87, and all but two were greater than 0.90. A α result, the standard deviation of the residuals from each regression is quite small and hence so is the standard error of estimate of α, and this provides the main advantage of grouping in these tests.

Table 3

Summary of Coefficients for the Subperiods

Item*	Subperiod†	Portfolio Number 1	2	3	4	5	6	7	8	9	10	M_M
$\hat{\beta}$	1	1.5416	1.3993	1.2620	1.1813	1.0750	0.9197	0.8569	0.7510	0.6222	0.4843	1.0000
	2	1.7157	1.3196	1.1938	1.0861	0.9697	0.9254	0.8114	0.7675	0.6647	0.5626	1.0000
	3	1.5427	1.3598	1.1822	1.1216	1.0474	0.9851	0.9180	0.7714	0.6547	0.4868	1.0000
	4	1.4423	1.2764	1.1818	1.0655	0.9957	0.9248	0.8601	0.7800	0.6614	0.6226	1.0000
$\hat{\alpha} \bullet 10^2$	1	0.7366	0.1902	0.3978	0.1314	-0.0650	-0.0501	-0.2190	-0.3786	-0.2128	-0.0710	
	2	-0.2197	-0.1300	-0.1224	0.0653	-0.0805	0.0914	0.1306	0.0760	0.2685	0.1478	
	3	-0.4614	-0.3994	-0.1189	0.0052	0.0002	-0.0070	0.1266	0.2428	0.3032	0.2035	
	4	-0.4475	-0.2536	-0.2329	-0.0654	0.0840	0.1356	0.1218	0.3257	0.3338	0.3685	
$t(\hat{\alpha})$	1	1.3881	0.6121	1.4037	0.6484	-0.3687	-0.1882	-1.0341	-1.7601	-0.7882	-0.1978	
	2	-0.4256	-0.7605	-0.8719	0.5019	-0.6288	0.8988	1.1377	0.6178	1.7853	0.8377	
	3	-2.9030	-3.6760	-1.5160	0.0742	0.0029	-0.1010	1.8261	3.3768	3.3939	1.9879	
	4	-2.8761	-2.4603	-2.7886	-0.7722	1.1016	1.7937	1.6769	3.8772	3.0651	3.2439	
$\overline{R}$	1	0.0412	0.0326	0.0317	0.0272	0.0230	0.0197	0.0166	0.0127	0.0115	0.0099	0.0220
	2	0.0233	0.0183	0.0165	0.0168	0.0136	0.0147	0.0134	0.0122	0.0126	0.0098	0.0149
	3	0.0126	0.0112	0.0120	0.0126	0.0117	0.0109	0.0115	0.0110	0.0103	0.0075	0.0112
	4	0.0082	0.0082	0.0081	0.0087	0.0096	0.0095	0.0088	0.0101	0.0092	0.0092	0.0088
σ	1	0.2504	0.2243	0.2023	0.1886	0.1715	0.1484	0.1377	0.1211	0.1024	0.0850	0.1587
	2	0.1187	0.0841	0.0758	0.0690	0.0618	0.0586	0.0519	0.0494	0.0441	0.0392	0.0624
	3	0.0581	0.0505	0.0436	0.0413	0.0385	0.0364	0.0340	0.0289	0.0253	0.0203	0.0363
	4	0.0577	0.0503	0.0463	0.0420	0.0391	0.0365	0.0340	0.0312	0.0277	0.0265	0.0386

* $\overline{R}_M$ = average monthly excess returns, σ = standard deviation of the monthly excess returns.

†Subperiod 1 = January, 1931-September, 1939; 2 = October, 1939-June, 1948; 3 =July, 1948-March, 1957; 4 = April, 1957-December, 1965.

III. Cross-sectional Tests of the Model

A. Tests of the Two-Factor Model.

Although the time series tests discussed in Section II provide a test of the traditional form of the asset pricing model, they cannot be used to test the two-factor model directly. The cross-sectional tests, however, do furnish an opportunity to test the linearity of the relation between returns and risk implied by (2) or $(2')$ without making any explicit specification of the intercept. Recall that the traditional form of the model implies $\gamma_0 = 0$ and $\gamma_1 = \tilde{R}_M$. The two factor model merely requires the linearity of (2) to hold for any specific cross section and allows the intercept to be nonzero. At this level of specification we shall not specify the size or even the sign of γ_0. We shall be able to make some statements on this point after a closer examination of the theory. However, we shall first examine the empirical evidence to motivate that discussion.

B. Measurement Errors and Bias in Cross-sectional Tests.

We consider here the problems caused in cross-sectional tests of the model by measurement errors in the estimation of the security risk measures.[8] Let β_j represent the true (and unobservable) systematic risk of firm *j* and $\hat{\beta}_j = \beta_j + \tilde{\in}_j$ be the measured value of the systematic risk of firm *j* where we assume that $\tilde{\in}_j$, the measurement error, is normally distributed and for all *j* satisfies

$$E\left(\tilde{\in}_j\right) = 0 \tag{7a}$$

8 See also Miller and Scholes (1972), who provide a careful analysis (using procedures that are complementary to but much different from those suggested here) of many of these problems with cross-sectional tests and their implications for the interpretation of previous empirical work.

$$E\left(\tilde{\in}_j \beta_j\right) = 0 \tag{7b}$$

$$E\left(\tilde{\in}_j \tilde{\in}_i\right) = \begin{cases} 0 & i \neq j \\ \sigma^2\left(\tilde{\in}\right) & i = j \end{cases} \tag{7c}$$

The traditional form of the asset pricing model and the assumptions of the market model imply that the mean excess return on a security

$$\overline{R}_j = \frac{\sum_{t=1}^{T} \tilde{R}_{jt}}{T} \tag{8}$$

observed over T periods can be written as

$$\overline{R}_j = E\left(\overline{R}_j \middle| \overline{R}_M\right) + \overline{e}_j = \overline{R}_M \beta_j + \overline{e}_j \tag{9}$$

where $\overline{R}_M = \sum_{t=1}^{T} \tilde{R}_{Mt}/T$, $\overline{e}_j = \sum_{t=1}^{T} \overline{e}_{jt}/T$. Now an obvious test of the traditional form of the asset pricing model is to fit

$$\tilde{R}_j = \gamma_0 + \gamma_1 \hat{\beta}_j + \overline{e}_j^{\circ} \tag{10}$$

to a cross section of firms (where $\hat{\beta}_j$ is the estimated risk coefficient for each firm and $\overline{e}_j^{\circ} = \overline{e}_j - \gamma_1 \tilde{\in}_j$) and test to see if, as implied by the theory

$$\gamma_0 = 0 \quad \text{and} \quad \gamma_1 = \tilde{R}_M$$

There are two major difficulties with this procedure; the first involves bias due to the measurement errors in $\hat{\beta}_j$, and the second involves the apparent inadequacy of (9) as a specification of the process generating the data. The two-factor asset pricing model given by $\left(2'\right)$ implies that γ_0 and γ_1 are random coefficients-that is, in addition to the theoretical values above, they involve a variable that is random through time. If the two-factor model is the true model, the usual significance tests on γ_0 and γ_1 are misleading, since the data from a given cross section cannot provide any evidence on the standard deviation of $\tilde{r}_Z$ and hence results in a serious underestimate of the sampling error of $\hat{\gamma}_0$ and $\hat{\gamma}_1$. Ignoring this second difficulty for the moment, we shall first consider the measurement error problems and the cross-sectional empirical evidence. The random coefficients issue and appropriate significance tests in the context of the two-factor model are discussed in more detail in Section IV.

As long as the $\hat{\beta}_j$ contain the measurement errors $\tilde{\in}_j$, the least-squares estimates $\hat{\gamma}_0$ and $\hat{\gamma}_1$ in (10) will be subject to the well-known errors in variables bias and will be inconsistent, (cf. Johnston (1963, Chap. VI)). That is, assuming that $\tilde{e}_j$ and $\tilde{\in}_j$ are independent and are independent of the β_j in the cross-sectional sample,

$$p\,lim\,\hat{\gamma}_1 = \frac{\gamma_1}{1+\sigma^2(\tilde{\in})/S^2(\beta_j)} \tag{11}$$

where $S^2(\beta_j)$ is the cross-sectional sample variance of the true risk parameters β_j. Even for large samples, then, as long as the variance of the errors in the risk measure $\sigma^2(\tilde{\in})$ is positive, the estimated coefficient $\hat{\gamma}_1$ will be biased toward zero and $\hat{\gamma}_0$ will therefore be biased away from zero. Hence tests of the significance of the differences $\hat{\gamma}_0 - 0$ and $\hat{\gamma}_1 - \tilde{R}_M$ will be misleading.

C. The Grouping Solution to the Measurement Error Problem.

We show in the Appendix that by appropriate grouping of the data to be used in estimating (10) one can substantially reduce the bias introduced through the existence of measurement errors in the $\hat{\beta}_j$. In essence the procedure amounts to systematically ordering the firms into groups (in fact by the same procedure that formed the ten portfolios used in the time series tests in Section II) and then calculating the risk measures $\hat{\beta}$ for each portfolio using the time series of portfolio returns. This procedure can greatly reduce the sampling error in the estimated risk measures; indeed, for large samples and independent errors, the sampling error is virtually eliminated. We then estimate the cross-sectional parameters of (10) using the portfolio mean returns over the relevant holding period and the risk coefficients obtained from estimation of (6) from the time series of portfolio returns. If appropriate grouping procedures are employed, this procedure will yield consistent estimates of the parameters γ_0 and γ_1 and thus will yield virtually unbiased estimates for samples in which the number of securities entering each group is large. Thus, by applying the cross-sectional test to our ten portfolios rather than to the underlying individual securities, we can virtually eliminate the measurement error problem.[9]

D. The Cross-sectional Empirical Results.

Given the 35 years of monthly returns on each of the ten portfolios calculated as explained in Section II, we then estimated $\hat{\beta}_K$ and $\overline{R}_K (K = 1, 2, \ldots, 10)$ for each portfolio, using all 35 years of monthly data. These estimates (see Table 2) were then used in estimating the cross-sectional relation given by (10) for various holding periods.

---

9 Intuitively one can see that the measurement error problem is virtually eliminated by these procedures because the errors in $\hat{\beta}_K$ become extremely small. Since the correlations $r(\tilde{R}_K, \tilde{R}_M)$ are so high in Table 2, the standard errors of estimate of the coefficients β_K are all less than 0.022, and nine of them are less than 0.012. The average standard error of estimate for the ten $\hat{\beta}_K$ coefficients given in Table 2 for the entire period was 0.0101 and the cross-sectional variance of the $\hat{\beta}_K, S^2(\hat{\beta}_K)$ was 0.1144. Hence, assuming $S^2(\hat{\beta}_K) = S^2(\beta_K)$, squaring 0.0101, and using (11), we see that our estimate of γ_1 will be greater than 99.9% of its true value.

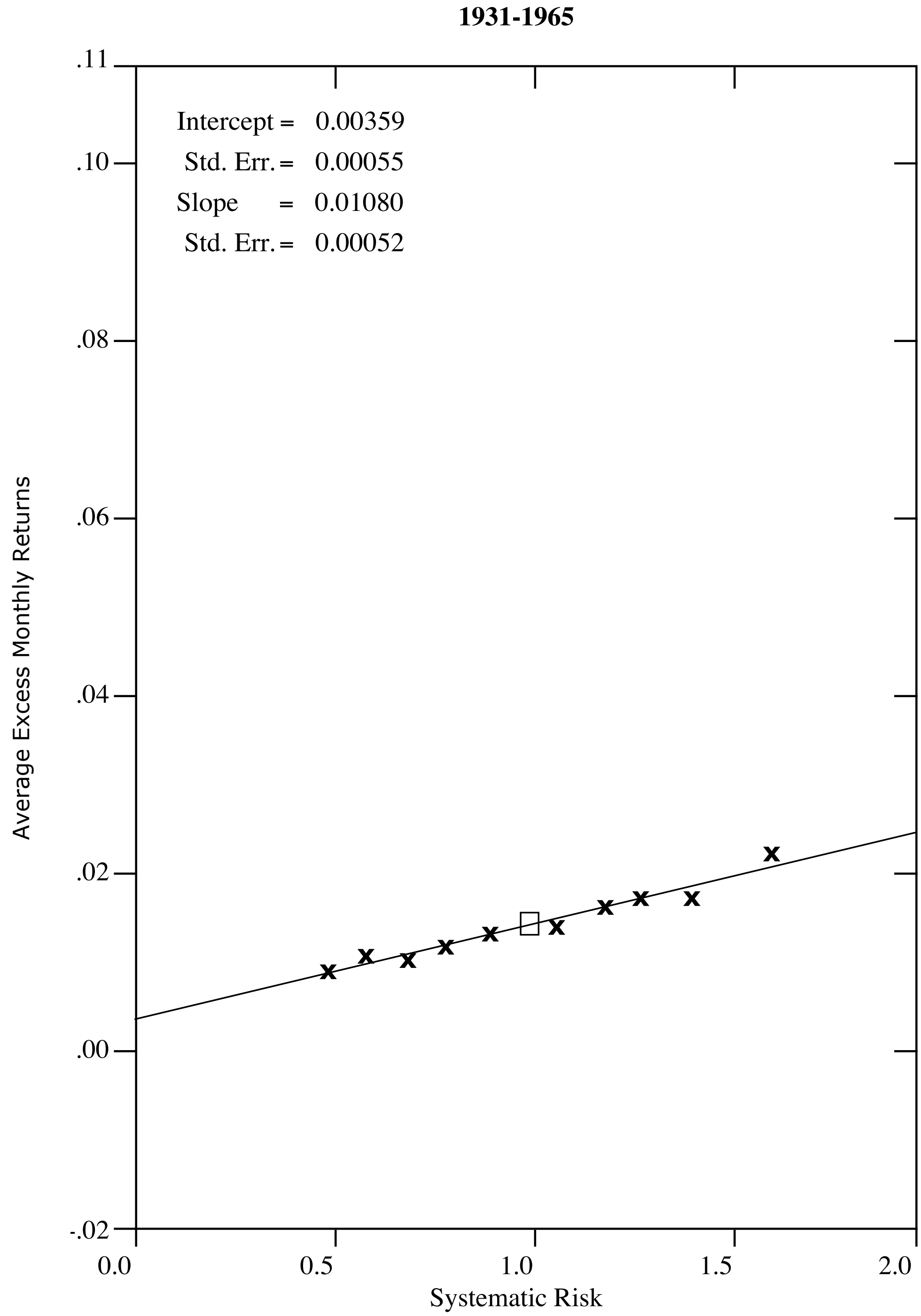

Figure 1 Average excess monthly returns versus systematic risk for the 35-year period 1931-65 for each of ten portfolios (denoted by ×) and the market portfolio (denoted by □).

Figure 1 is a plot of $\overline{R}_K$ versus $\hat{\beta}_K$ for the 35-year holding period January, 1931-December, 1965. The symbol × denotes the average monthly excess return and risk of each of the ten portfolios. The symbol □ denotes the average excess return and risk of the market portfolio (which by the definition of β is equal to unity). The line represents the least squares estimate of the relation between $\overline{R}_K$ and $\hat{\beta}_K$. The "intercept" and "slope" (with their respective standard errors given in parentheses) in the upper portion of the figure are the coefficients γ_0 and γ_1 of (10).

The traditional form of the asset pricing model implies that the intercept γ_0 in (10) should be equal to zero and the slope γ_1 should be equal to $\overline{R}_M$, the mean excess return on the market portfolio. Over this 35-year period, the average monthly excess return on the market portfolio $\overline{R}_M$, was 0.0142, and the theoretical values of the intercept and slope in Figure 1 are

$$\gamma_0 = 0 \qquad \text{and} \qquad \gamma_1 = 0.0142$$

The "t" values

$$t(\hat{\gamma}_0) = \frac{\hat{\gamma}_0}{s(\hat{\gamma}_0)} = \frac{0.00359}{0.00055} = 6.52$$

$$t(\hat{\gamma}_1) = \frac{\gamma_1 - \hat{\gamma}_1}{s(\hat{\gamma}_1)} = \frac{0.0142 - 0.0108}{0.00052} = 6.53$$

seem to indicate the observed relation is significantly different from the theoretical one. However, as we shall see, because (9) is a misspecification of the process generating the data, these tests vastly overstate the significance of the results.

We also divided the 35-year interval into four equal subperiods, and Figures 2 through 5 present the plots of the $\overline{R}_K$ versus the $\hat{\beta}_K$ for each of these intervals. In order to obtain better estimates of the risk coefficients for each of the subperiods, we used the coefficients previously estimated over the entire 35-year period.[10] The graphs indicate that the relation between return and risk is linear but that the slope is related in a non-stationary way to the theoretical slope for each period. Note that the traditional model implies that the theoretical relationship (not drawn) always passes through the two points given by the origin (0, 0) and the average market excess returns represented by □ in each figure. In the first sub-period (see Fig. 2) the empirical slope is steeper than the theoretical slope and then becomes successively flatter in each of the following three periods. In the last subperiod (see Fig. 5) the slope $\hat{\gamma}_1$ even has the "wrong" sign.

TABLE 4

Summary of Cross-sectional Regression Coefficients and Their *t* Values

	Time Period				
	Total Period	*Subperiods*			
	1/31-12/65	1/31-9/39	10/39-6/48	7/48-3/57	4/57-12/65
$\hat{\gamma}_0$	0.00359	-0.00801	0.00439	0.00777	0.01020
$\hat{\gamma}_1$	0.0108	0.0304	0.0107	0.0033	-0.0012
$\gamma_1 = \overline{R}_M$	0.0142	0.0220	0.0149	0.0112	0.0088
$t(\hat{\gamma}_0)$	6.52	-4.45	3.20	7.40	18.89
$t(\gamma_1 - \hat{\gamma}_1)$	6.53	-4.91	3.23	7.98	19.61

10 The analysis was also performed where the coefficients were estimated for each subperiod, and the results were very similar because the $\hat{\beta}_\kappa$ were quite stable over time. We report these results since this estimation procedure seemed to result in a slightly larger spread of the $\hat{\beta}_\kappa$ and since the increased sample sizes tends to further reduce the bias caused by the variance of the measurement error in $\hat{\beta}_\kappa$.

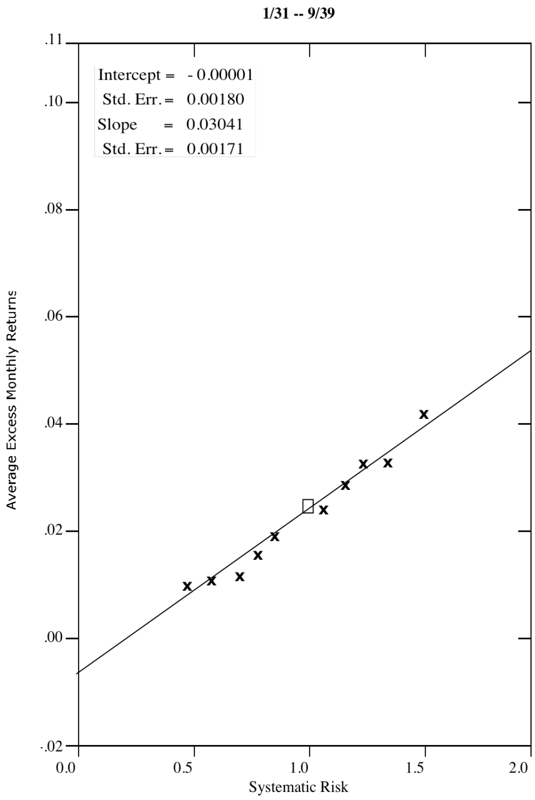

Figure 2 Average Excess monthly returns versus systematic risk for the 105 -month period January, 1931-September, 1939. Symbols as in Figure 1.

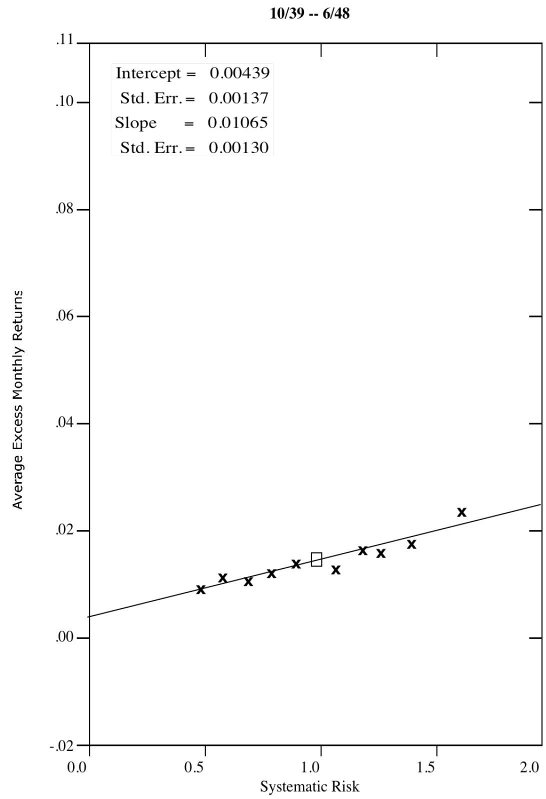

Figure 3 Average Excess monthly returns versus systematic risk for the 105 -month period October, 1939-June, 1948. Symbols as in Figure 1.

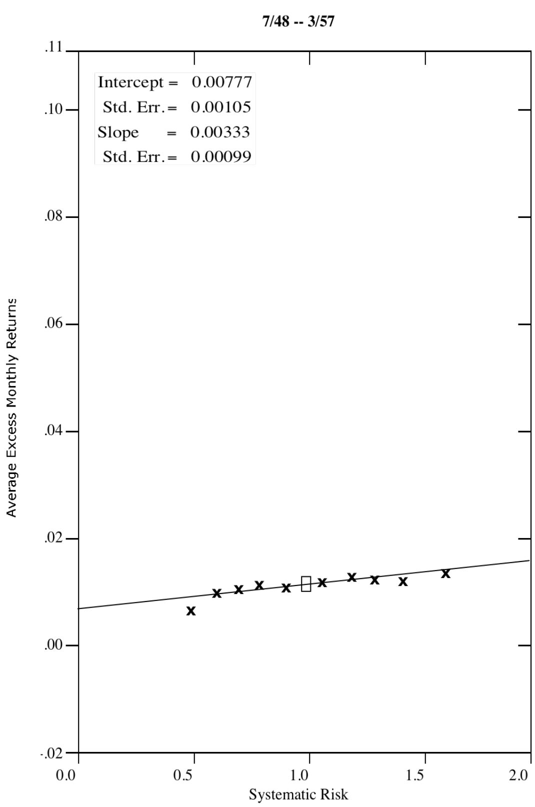

Figure 4 Average Excess monthly returns versus systematic risk for the 105 -month period July, 1948-March, 1957. Symbols as in Figure 1.

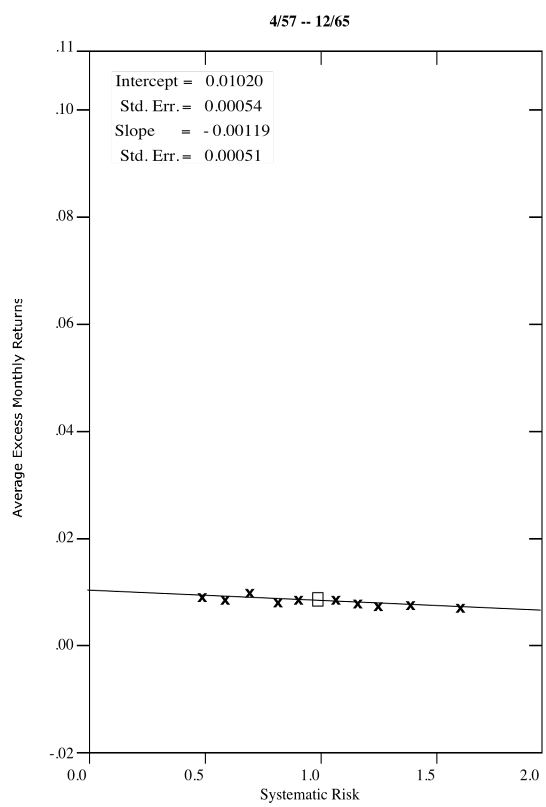

Figure 4 Average Excess monthly returns versus systematic risk for the 105 -month period April, 1957 - December, 1965. Symbols as in Figure 1.

The coefficients $\hat{\gamma}_0$, $\hat{\gamma}_1$, γ_1 and the "*t*" values of $\hat{\gamma}_0$ and $\gamma_1 - \hat{\gamma}_1$ are summarized in Table 4 for the entire period and for each of the four subperiods. The smallest "*t*" value given there is 3.20, and all seem to be "significantly" different from their theoretical values. However, as we have already maintained, these "*t*" values are somewhat misleading because the estimated coefficients fluctuate far more in the subperiods than the estimated sampling errors indicate. This evidence suggests that the model given by (9) is misspecified. We shall now attempt to deal with this specification problem and to furnish an alternative formulation of the model.

IV. A Two-Factor Model

A. Form of the Model.

As mentioned in the introduction, Black [1970] has shown under assumptions identical to that of the asset pricing model that, if riskless borrowing opportunities do not exist, the expected return on any asset *j* will be given by

$$E(\tilde{r}_j) = E(\tilde{r}_z)(1-\beta_j) + E(\tilde{r}_M)\beta_j \tag{12}$$

where $\tilde{r}_z$ represents the return on a "zero beta" portfolio--a portfolio whose covariance with the returns on the market portfolio $\tilde{r}_M$ is zero.[11]

Close examination of the empirical evidence from both the cross-sectional and the time series tests indicates that the results are consistent with a model that expresses the return on a security as a linear function of the market factor r_M, (with a coefficient of β_j) and a second factor r_z, (with a coefficient of $1-\beta_j$) The function is

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11 In fact, there is an infinite number of such zero β portfolios. Of all such portfolios. however, r_z is the return on the one with minimum variance. (We are indebted to John Long for the proof of this point.)

$$\tilde{r}_{jt} = \tilde{r}_{Zt}\left(1-\beta_j\right) + \tilde{r}_{Mt}\beta_j + \tilde{w}_{jt} \tag{13}$$

Because the coefficient of the second factor is a function of the security's β, we call this factor the beta factor. For a given holding period T, the average value of $\tilde{r}_{Zt}$ will determine the relation between $\hat{\alpha}$ and $\hat{\beta}$ for different securities or portfolios. If the data are being generated by the process given by (13) and if we estimate the single variable time series regression given by (6), then the intercept $\hat{\alpha}$ in that regression will be

$$\hat{\alpha} = \left(\bar{r}_Z - \bar{r}_F\right)\left(1-\hat{\beta}_j\right) = \overline{R}_Z\left(1-\hat{\beta}_j\right) \tag{14}$$

where $\bar{r}_Z = \sum_{t=1}^{T} \tilde{r}_{Zt}/T$ is the mean return on the beta factor over the period, $\bar{r}_F$ is the mean risk-free rate over the period, and $\overline{R}_Z$ is the difference between the two. Thus if $\overline{R}_Z$ is positive, high-beta securities will tend to have negative $\hat{\alpha}$'s, and low-beta securities will tend to have positive $\hat{\alpha}$'s. If $\overline{R}_Z$ is negative, high-beta securities will tend to have positive $\hat{\alpha}$'s, and low-beta securities will tend to have negative $\hat{\alpha}$'s.

In addition, if we estimate the cross-sectional regression given by (10), the expanded two-factor model implies that the true values of the parameters γ_0 and γ_1 will not be equal to zero and $\overline{R}_M$ but instead will be given by

$$\gamma_0 = \overline{R}_Z \qquad \text{and} \qquad \gamma_1 = \overline{R}_M - \overline{R}_Z$$

Hence if $\overline{R}_Z$ is positive, γ_0 will be positive and γ_1 will be less than $\overline{R}_M$. If $\overline{R}_Z$ is negative, γ_0 will be negative and γ_1 will be greater than $\overline{R}_M$.

Thus we can interpret Table 3 and Figures 2 through 5 as indicating that $\overline{R}_Z$ was negative in the first subperiod and became positive and successively larger in each of the following subperiods.

Examining (12), we see that the traditional form of the capital asset pricing model, as expressed in (1), is consistent with the present two-factor model if

$$E(\tilde{R}_Z) = 0 \tag{15}$$

and (questions of statistical efficiency aside) any test for whether α_K for a portfolio is zero is equivalent to a test for whether $E(\tilde{R}_Z)$ is zero. The results in Table 3 suggest that $E(\tilde{R}_Z)$ is not stationary through time. For example, α_K for the lowest risk portfolio (number 10) is negative in the first subperiod and positive in the last subperiod, with a "*t*" value of 8. Thus it is unlikely that the true values of α_K were the same in the two subperiods (each of which contains 105 observations) and thus unlikely that the true values of $E(R_Z)$ were the same in the two subperiods, and we shall derive formal tests of this proposition below.

The existence of a factor $\overline{R}_Z$ with a weight proportional to $1-\beta_j$ in most securities is also suggested by the unreasonably high "*t*" values[12] obtained in the cross-sectional regressions, as given in Table 4. Since γ_0 and γ_1 involve $\overline{R}_Z$, which is a random variable from cross section to cross section, and since no single cross-sectional run can provide any information whatsoever on the variability of $\overline{R}_Z$, this element is totally ignored in the usual calculation of the standard errors of γ_0 and γ_1. It is not surprising, therefore, that each individual cross-sectional result seems so highly significant but so totally different from any other cross-sectional relationship. Of course the presence of infinite-variance stable distributions will also contribute to this type of phenomenon.

In addition, in an attempt to determine whether the linearity observed in Figures 1 through 5 was in some way due to the averaging involved in the long periods presented there, we replicated those plots for our ten portfolios for 17 separate two-year periods from 1932 to 1965. These results, which also exhibit a remarkable linearity, are presented in Figures 6a and 6b. Since the evidence seems to indicate that the all-risky asset model describes the data better than the traditional model, and since the definition of our "riskless" interest rate was somewhat arbitrary in any case, these plots were derived from calculations on the raw return data with no reference whatsoever to the "risk-free" rate defined earlier (including the recalculation of the ten portfolios and the estimation of the β_j). Figures 7 through 11 contain a replication of Figures 1 through 5 calculated on the same basis. These results indicate that the basic findings summarized previously cannot be attributed to misspecification of the riskless rate.

In summary, then, the empirical results suggest that the returns on different securities can be written as a linear function of two factors as given in (13), that the expected excess return on the beta factor $\tilde{R}_Z$ has in general been positive, and that the expected return on the beta factor has been higher in more recent subperiods than in earlier subperiods.

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[12] We say unreasonably high because the coefficients change from period to period by amounts ranging up to almost seven times their estimated standard errors.

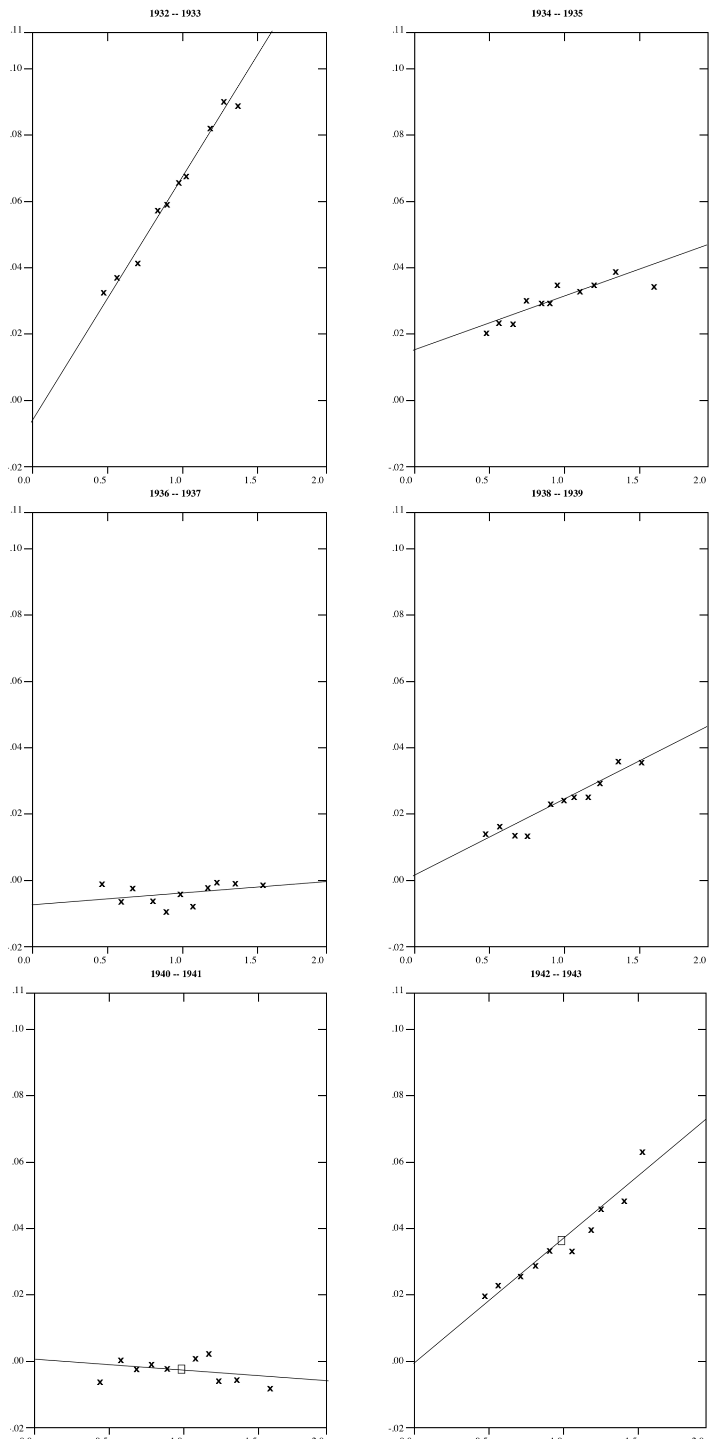


Figure 6 Average monthly returns versus systematic risk for 17 non-overlapping two-year periods from 1932 to 1965

Figure 6 Continued

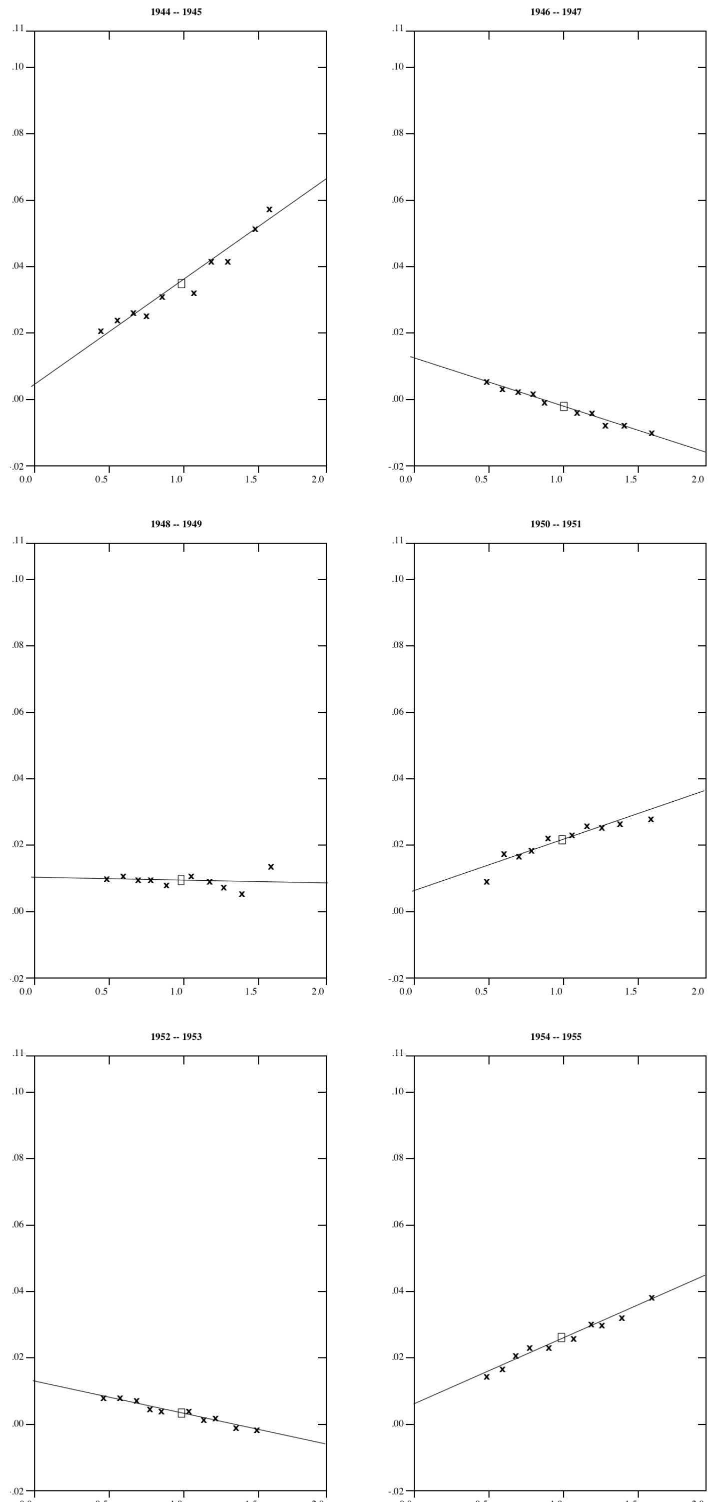
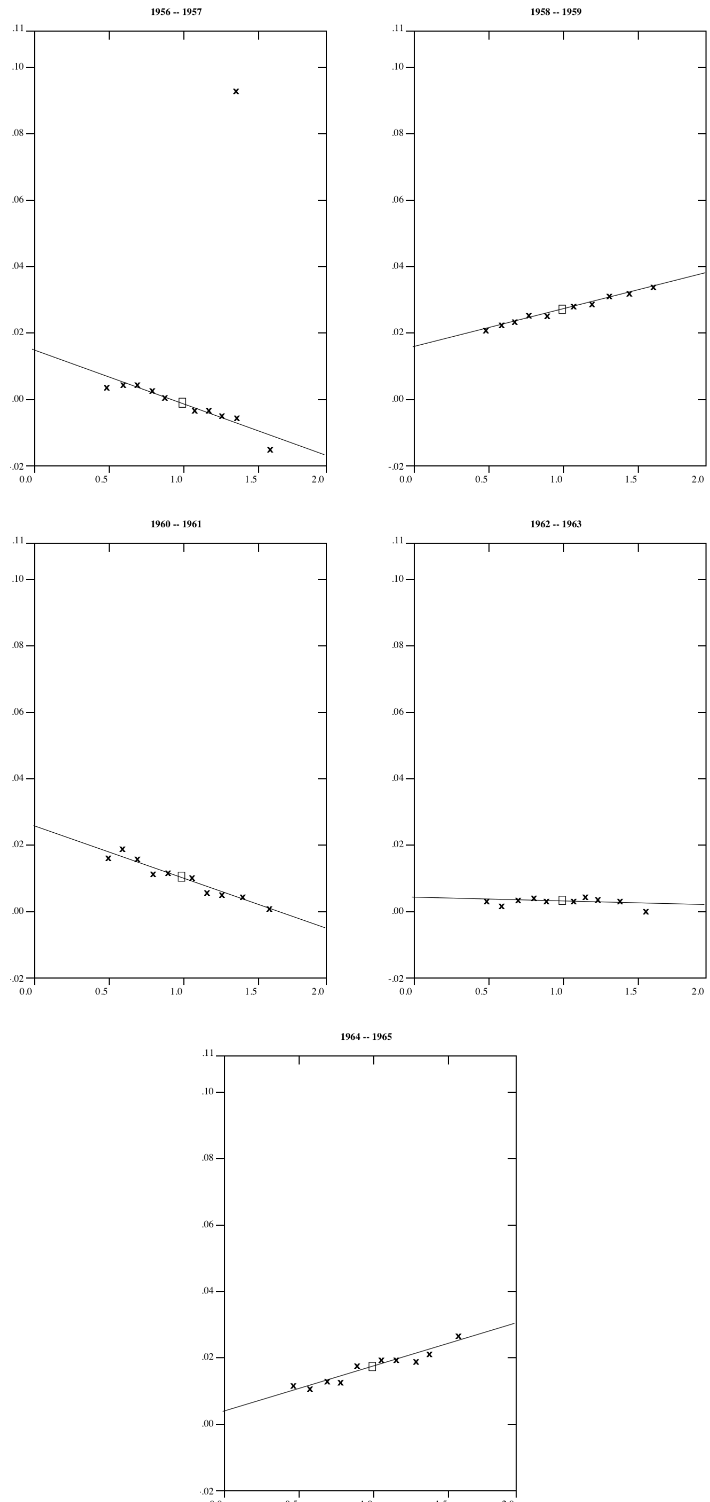


Figure 6 Continued



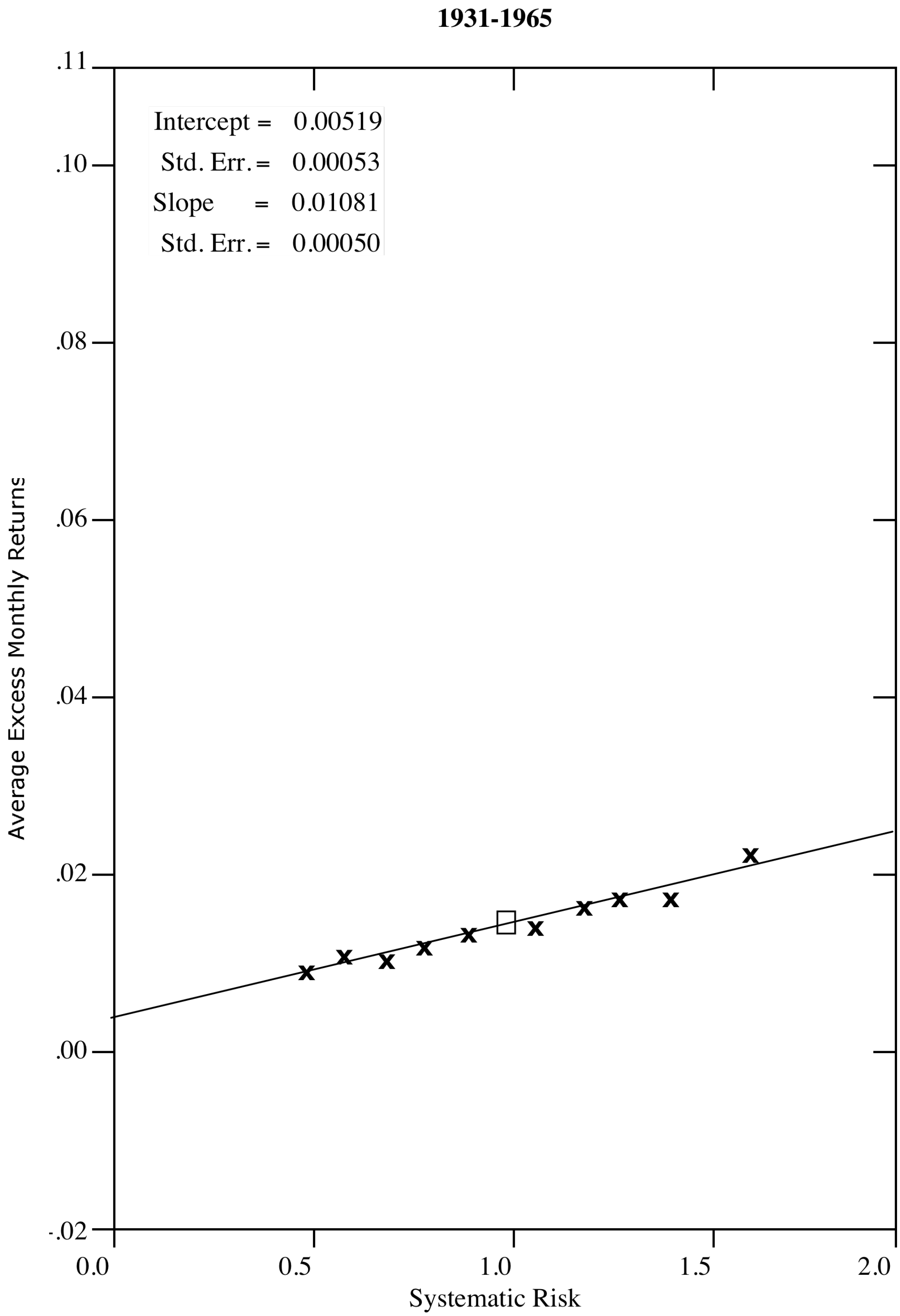


Figure 7 Average monthly returns versus systematic risk for the 35-year period 1931-65 for the ten portfolios and the market portfolio.

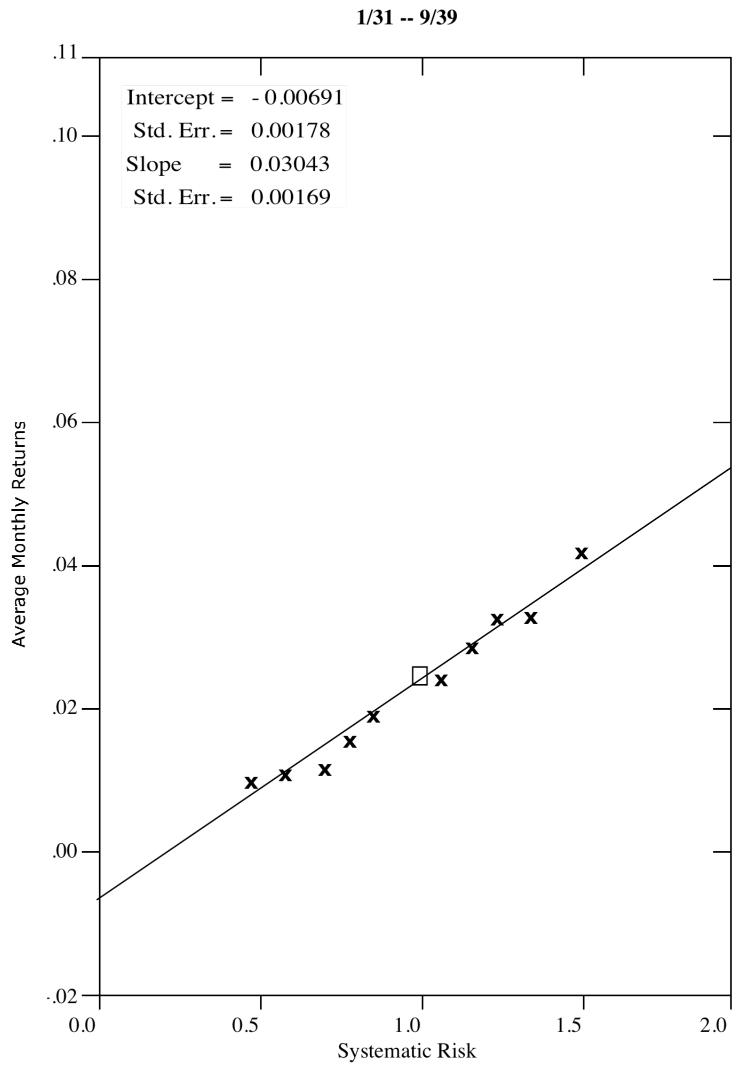


Figure 8 Average Excess monthly returns versus systematic risk for the 105 -month period January, 1931-September, 1939.

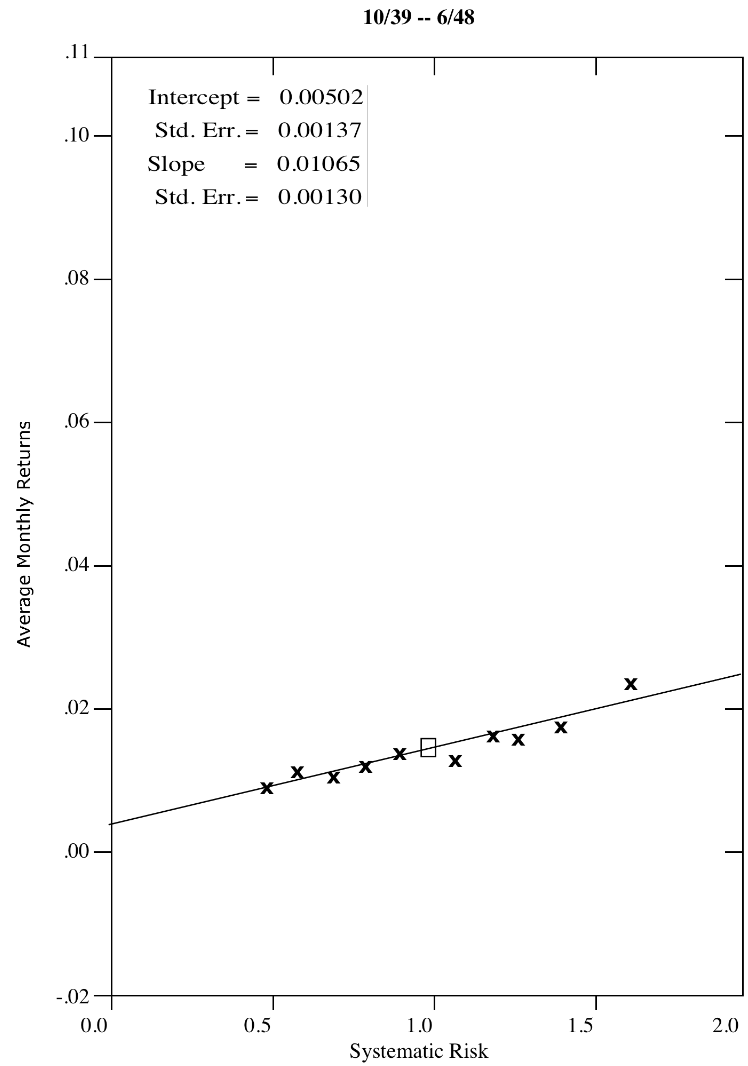


Figure 9 Average Excess monthly returns versus systematic risk for the 105 - month period October, 1939-June, 1948.

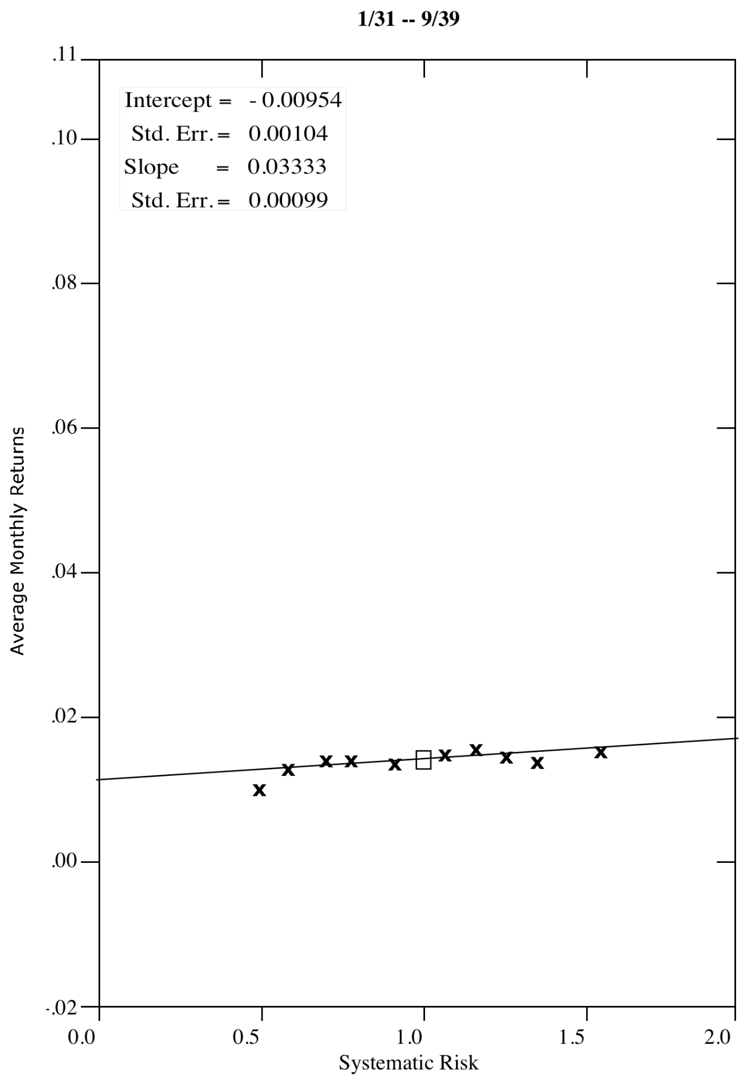


Figure 10 Average monthly returns versus systematic risk for the 105-month period July 1948-March, 1957.

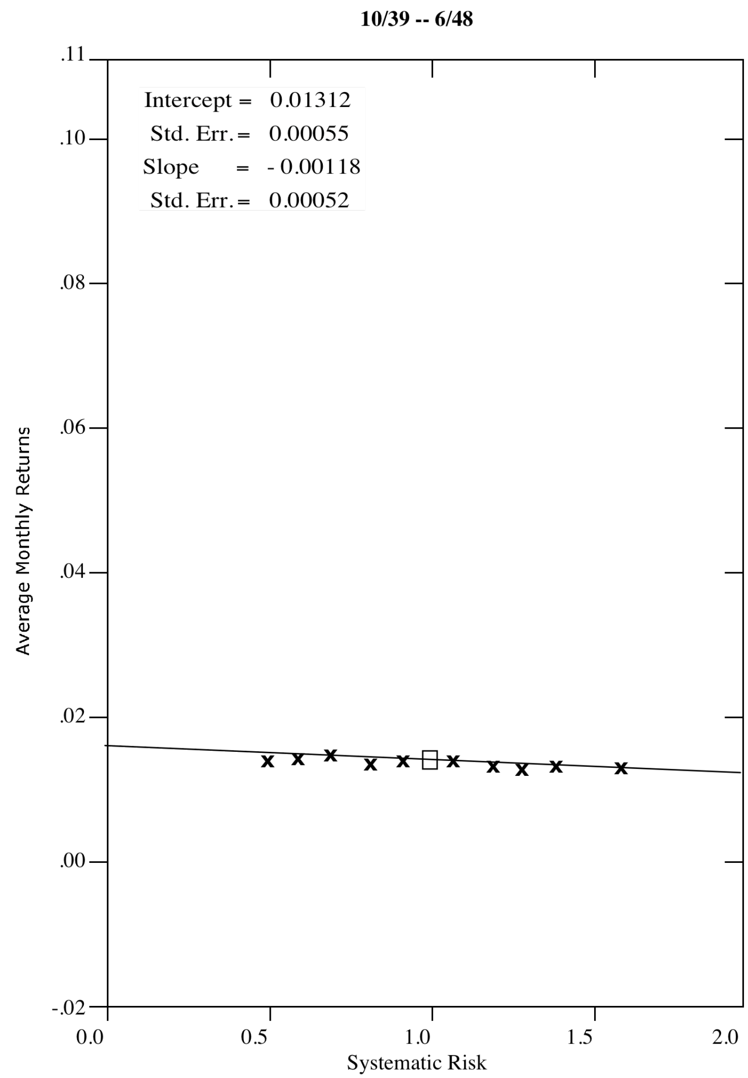


Figure 11 Average monthly returns versus systematic risk for the 105-month period April, 1957-December, 1965.

B. Explicit Estimation of the Beta Factor and a Crucial Test of the Model.

Since the traditional form of the asset pricing model is consistent with the existence of the beta factor as long as the excess returns on the beta factor have zero mean,[13] our purpose here is to provide a procedure for explicit estimation of the time series of the factor. Given such a time series, we can then make explicit estimates of the significance of its mean excess return rather than depending mainly on an examination of the $\hat{\alpha}_j$ for high- and low-beta securities. Solving (13) for $\tilde{r}_{Zt}$ plus the error term, we have an estimate $\hat{r}_{Zjt}$, of $\tilde{r}_{Zt}$

$$\hat{r}_{Zjt} = \frac{1}{\left(1-\beta_j\right)}\left[\tilde{r}_j - \beta_j \tilde{r}_{Mt}\right] = \tilde{r}_{Zt} + \tilde{u}_{jt} \tag{16}$$

where $\tilde{u}_{jt} = \tilde{w}_{jt} \big/ \left(1-\beta_j\right)$. We subscript $\hat{r}_{Zjt}$ by j to denote that this is an estimate of $\tilde{r}_{Zt}$ obtained from the jth asset or portfolio. Now, since we can obtain as many separate estimates of $\tilde{r}_{Zt}$ as we have securities or portfolios, we can formulate a combined estimate

$$r^*_{Zt} = \sum_j h_j \hat{r}_{Zjt} \tag{17}$$

which is a linear combination of the $\hat{r}_{Zjt}$, to provide a much more efficient estimate of $\tilde{r}_{Zt}$. The problem is to find that linear combination of the $\hat{r}_{Zjt}$ which minimizes the error variance in the estimate of $\tilde{r}_{Zt}$. That is, we want to

13 Although the traditional form of the model is consistent with the existence of the β factor if its excess return had a zero mean, clearly it would not provide as complete an explanation of the structure of asset returns as a model that explicitly incorporated such a factor. In particular, under these circumstances the traditional form would provide an adequate description of security returns over fairly lengthy periods of time, say three years or more, but it would probably not furnish an adequate description of security returns over much shorter intervals.

$$\min_{h_j} E\left(r^*_{Zt} - \tilde{r}_{Zt}\right)^2 = \min_{h_j} E\left(\sum_j h_j \hat{r}_{Zjt} - \tilde{r}_{Zt}\right)^2$$

subject to $\sum_j h_j = 1$, since we want an unbiased estimate. From the Lagrangian we obtain the first-order conditions

$$h_j \sigma^2(\tilde{u}_j) - \lambda = 0 \qquad j = 1, 2, \ldots, N \tag{18}$$

where λ is the Lagrangian multiplier and N is the total number of securities or non-overlapping portfolios. These conditions imply that

$$\frac{h_j}{h_i} = \frac{\sigma^2(\tilde{u}_i)}{\sigma^2(\tilde{u}_j)} \quad \text{for all } i \text{ and } j \tag{19}$$

which implies that the optimal weights h_j are proportional to $1/\sigma^2(\tilde{u}_j)$. That is,

$$h_j = \frac{K}{\sigma^2(\tilde{u}_j)} \qquad j = 1, 2, \ldots, N \tag{20}$$

where $K = 1/\sum_j \left[1/\sigma^2(\tilde{u}_j)\right]$ is a normalizing constant. But from the definition of $\tilde{u}_j$ we know that $\sigma^2(\tilde{u}_j) = \sigma^2(\tilde{w}_j)/(1-\beta_j)^2$, so

$$h_j = \frac{K(1-\beta_j)^2}{\sigma^2(\tilde{w}_j)} \tag{21}$$

Equation (21) makes sense, for we are then weighting the estimates in proportion to $(1-\beta_j)^2$ and inversely proportional to $\sigma^2(\tilde{w}_j)$. However, since we cannot observe $\sigma^2(\tilde{w}_j)$ directly,[14] we are forced, for lack of explicit estimates, to assume that the $\sigma^2(\tilde{w}_j)$ are all identical and to use as our weights

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14 We only observe the residual variance from the single variable regression, and, as we can see from (13), this will be equal to $(1-\beta_j)^2 \sigma^2(\tilde{r}_z) + \sigma^2(\tilde{w}_j)$. However, there are more general procedures for estimating

$$h_j = K'\left(1-\beta_j\right)^2 \tag{22}$$

where $K' = 1\Big/\sum_j \left(1-\beta_j\right)^2$.

Equations (17) and (22) thus provide an unbiased and (approximately) efficient procedure for estimating $\tilde{r}_{Zt}$ utilizing all available information. However, there is a problem of bias involved in actually applying this procedure to the security data. The coefficient β_j is of course unobservable, and in general if we use our estimates $\hat{\beta}_j$ in the weighting procedure we will introduce bias into our estimate of $\tilde{r}_{Zt}$. To understand this, recall that $\hat{\beta}_j = \beta_j + \in_j$, substitute this into (13) with the necessary additions and subtractions, and solve for the estimate

$$\hat{r}_{Zjt} = \frac{\tilde{r}_{jt} - \hat{\beta}_j \tilde{r}_{Mt}}{\left(1-\hat{\beta}_j\right)} = \frac{\tilde{r}_{Zt}\left(1-\beta_j\right) + \tilde{w}_j - \in_j \tilde{r}_{Mt}}{\left(1-\hat{\beta}_j\right)}$$

Substituting this into (17), using (22), rearranging terms, and taking the probability limit, we have

$$\underset{N\to\infty}{p\,lim}\, r^*_{Zt} = \frac{C_t\left[S^2(\beta) + \left(1-\overline{\beta}\right)^2\right] + \sigma^2\left(\tilde{\in}\right) r_{Mt}}{\left[S^2(\beta) + \left(1-\overline{\beta}\right)^2\right] + \sigma^2\left(\tilde{\in}\right)} \tag{23}$$

where $S^2(\beta)$ is the cross-sectional variance of the β_j and $\overline{\beta}$ is the mean. However, the average standard deviation of the measurement error $\sigma\left(\tilde{\in}_j\right)$ for our portfolios is only

$\tilde{r}_{Zt}$, in the situation of non identical $\sigma^2(\tilde{w}_j)$ and $cov(\tilde{w}_j,\tilde{w}_i) = 0$ for $j \neq i$. But we leave an investigation of the properties of these estimates and some additional tests of the two-factor model for a future paper. If the assumption of identical $\sigma^2(\tilde{w}_j)$ made here is inappropriate, we still obtain an unbiased estimate of the $\tilde{R}_z$. However, the estimated variance of $\tilde{R}_z$, which is of some interest, will be greater than the true variance.

0.0101 (implying an average variance on the order of 0.0001), and since $S^2(\beta)$ for our ten portfolios is 0.1144 and $\bar{\beta}$ = 1.007, this bias will be negligible and we shall ignore it.

To begin, let us apply the foregoing procedures to the excess return data to obtain an estimate of $\tilde{R}_{Zt} = \tilde{r}_{Zt} - r_{Ft}$, the excess return on the beta factor. Substituting R_{jt} for r_{jt} and R_{Mt} for r_{Mt} in (16), the $\hat{R}_{Zjt}$ were estimated for each of our ten log portfolios. These were then averaged to obtain the estimate

$$R^*_{Zt} = \sum_j h_j \hat{R}_{Zjt} = K' \sum_j \left(1-\beta_j\right)^2 \left[\frac{\tilde{R}_{jt} - \hat{\beta}_j R_{Mt}}{1-\hat{\beta}_j}\right]$$

for each month t. The average of the R^*_{Zt} for the entire period and for each of the four subperiods are given in Table 5, along with their t values. Table 5 also presents the serial correlation coefficients $r\left(R^*_{Zt}, R^*_{Z,t-1}\right)$.[15] Note that the mean value $\overline{R}^*_Z$ of the beta factor

TABLE 5

Estimated Mean Values and Serial Correlation of the Excess Returns on the Beta Factor over the Entire Periods and the Four Subperiods•

Period	$\overline{R}^*_Z$	$\sigma\left(R^*_Z\right)$	$t\left(\overline{R}^*_Z\right)$	$r\left(R^*_{Zt}, R^*_{Z,t-1}\right)$	$t(r)$
1/31-12/65	0.00338	0.0436	1.62	0.113	2.33
1/31-9/39	-0.00849	0.0641	-1.35	0.194	1.49
10/39-6/48	0.00420	0.0455	0.946	0.208	2.19
7/48-3/57	0.00782	0.0199	4.03	-0.181	-1.87
4/57-12/65	0.00997	0.0228	4.49	0.414	4.60

•The values of $t\left(\overline{R}^*_Z\right)$ were calculated under the assumption of normal distributions.

15 The serial correlation for the entire period appears significant. Indeed, the serial correlation in the last period, 0.414, seems very large and even highly significant, with a t value of 4.6. However, the coefficients in the earlier periods seem to border on significance but show an inordinately large amount of variability, thus indicating substantial nonstationarity.

over the whole period has a "t" value of only 1.64. However, as hypothesized earlier, it was negative in the first subperiod and positive and successively larger in each of the following subperiods. Moreover, in the last two subperiods its "t" values were 4.03 and 4.49, respectively. These results seem to us to be strong evidence favoring rejection of the traditional form of the asset pricing model which says that $\overline{R}_Z^*$ should be insignificantly different from zero.

In order to be sure that the significance levels reported in Table 5 are not spurious and due only to the misapplication of normal distribution theory to a situation in which the variables may actually be distributed according to the infinite variance members of the stable class of distributions. We have performed the significance tests using the stable distribution theory outlined by Fama and Roll (1968). Table 6 presents the standardized variates (i.e., the "t" values) for $\overline{R}_Z^*$ for each of the sample periods given in Table 5 along

Table 6

Normalized Variate [i.e., t Value $t\left(\overline{R}_Z^*,\alpha\right)=\overline{R}_Z^*/\sigma\left(\overline{R}_Z^*,\alpha\right)$] of the Excess Return on the Beta Factor Under the Assumption of Infinite Variance Symmetric Stable Distributions

	α					
Period	1.5	1.6	1.7	1.8	1.9	2.0
1/31-12/65	1.33	1.71	2.14	2.61	3.11°	3.65°
1/31-9/39	-1.11	-1.44	-1.71	-2.00	-2.29	-2.58
10/39-6/48	0.82	1.00	1.18	1.38	1.58	1.79
7/48-3/57	2.60	3.16	3.75°	4.37°	5.00°	5.66°
4/57-12/65	3.05	3.70	4.40°	5.11°	5.86°	6.63°
t Value at the 5% level of significance (two-tail)†	4.49	3.90	3.48	3.16	2.93	2.77

Note: α= characteristic exponent, $\sigma\left(\overline{R}_Z^*,\alpha\right)$ = dispersion parameter of the distribution.

†Cf. Fama and Roll (1968).

with the "t" values at the 5% level of significance (two-tail) under alternative assumptions regarding the value of α, the characteristic exponent of the distribution. The smaller is α, the higher are the extreme tails of the probability distribution; α = 2 corresponds to the normal distribution and α = 1 to the Cauchy distribution. Evidence presented by Fama [1965] seems to indicate that α is probably in the range 1.7 to 1.9 for common stocks. We have not attempted to obtain explicit estimates of α for our data, since currently known estimation procedures are quite imprecise and require extremely large samples (up to 2,000 observations). Therefore we have simply presented the "t" values calculated according to the procedures suggested by Fama and Roll (1968) for six values of α ranging from 1.5 to 2.0. The coefficients in Table 6 that are significant at the 5% level are noted with an asterisk. Clearly, if α is greater than 1.7, the results confirm the impression gained from the normal tests given in Table 5.

Note that the estimates in Table 5 and 6 were obtained from the excess return data; therefore, although the figures are of interest for testing traditional form of the model, they do not give the appropriate level of the mean value of $\tilde{r}_Z$. The estimates $\bar{r}_Z^*$ and $\bar{r}_M$ obtained from the total return data used in Figures 6 through 11 appear in Table7, along with $\sigma(\tilde{r}_Z^*)$ and $\sigma(\tilde{r}_M)$ and the estimated values of γ_0 and γ_1 for the cross-sectional regressions [given by (10)] for each of the various sample periods portrayed in Figures 6 through 11. (Recall that the two-factor model implies $\gamma_0 = \bar{r}_Z$ and $\gamma_1 = \bar{r}_M - \bar{r}_Z$.) One additional item of interest in judging the importance of the beta factor in the determination of security returns is its standard deviation relative to that of the market returns. As Table 7 reveals, $\sigma(\tilde{r}_Z^*)$ is roughly 50% as large as $\sigma(\tilde{r}_M)$. Comparison of $\bar{r}_Z^*$ and $\bar{r}_M$ in Table 7 for the four 105-month subperiods indicates that the mean returns on the beta factor were approximately equal to the average market returns in the last two periods covering the interval July, 1948-December, 1965. Apparently, then, the relative magnitudes of $\bar{r}_Z^*$ and $\bar{r}_M$ indicate that the beta factor is economically as well as statistically significant.

TABLE 7
Mean and Standard Deviation of Returns on the Zero Beta and Market Portfolios and the Cross-sectional Regression Coefficients [from (10)] for Various Sample Periods

Time Period	$\bar{r}_Z^*$	$\bar{r}_M$	$\bar{r}_M - \bar{r}_Z^*$	$\sigma(\tilde{r}_Z^*)$	$\sigma(\tilde{r}_M)$	γ_0 [a]	γ_1 [a]
1931-1965	0.004980	0.015800	0.010820	0.042584	0.089054	0.005190	0.010807
1/31-9/39	-0.007393	0.023067	0.030459	0.063927	0.158707	-0.006913	0.030429
10/39-6/48	0.004833	0.015487	0.010665	0.045520	0.062414	0.005021	0.010652
7/48-3157	0.009591	0.012915	0.003324	0.019895	0.036204	0.009537	0.003327
4/57-12/65	0.012889	0.011723	-0.001167	0.022631	0.038470	0.013115	-0.001181
1931	-0.047243	-0.037573	0.009669	0.040827	0.152924	-0.045492	0.009557
1932-1933	-0.009180	0.065574	0.074754	0.059741	0.245281	-0.008286	0.074696
1934-1935	0.015549	0.031250	0.015701	0.048551	0.097739	0.015542	0.015702
1936-1937	-0.007749	-0.004538	0.005211	0.032589	0.084786	-0.007336	0.003194
1938-1939	0.001919	0.024436	0.022517	0.100490	0.147129	0.001514	0.022543
1940-1941	-0.001308	-0.003902	-0.002596	0.043481	0.072454	-0.000646	-0.002638
1942-1943	-0.009898	0.035782	0.036780	0.066552	0.066451	-0.001069	0.036784
1944-1945	0.004511	0.036117	0.031507	0.032522	0.043560	0.004451	0.031517
1946-1947	0.010153	-0.002357	-0.013010	0.033074	0.056139	0.010946	-0.013061
1948-1949	0.009721	0.008529	-0.001192	0.019590	0.051471	0.009709	-0.001191
1950-1951	0.007163	0.020253	0.013090	0.028656	0.039764	0.007215	0.013087
1952-1953	0.012258	0.003054	-0.009204	0.014559	0.026896	0.012050	-0.009191
1954-1955	0.007432	0.027266	0.019834	0.019232	0.030804	0.007392	0.019836
1956-1957	0.010463	-0.003097	-0.013559	0.017638	0.032340	0.010555	-0.013565
1958-1959	0.014582	0.025060	0.011478	0.019982	0.028261	0.014205	0.011502
1960-1961	0.026825	0.010867	-0.015958	0.023178	0.036505	0.026753	-0.015953
1962-1963	0.004300	0.002728	-0.001571	0.026231	0.052144	0.005054	-0.001620
1964-1965	0.005032	0.017771	0.012738	0.014433	0.026761	0.005519	0.012707

[a] Cf. eq. (10).

V. Conclusion

The traditional form of the capital asset pricing model states that the expected excess return on a security is equal to its level of systematic risk, β, times the expected excess return on the market portfolio. That is, in capital market equilibrium, prices of assets adjust such that

$$E\left(\tilde{R}_j\right) = \gamma_1 \beta_j \tag{24}$$

where $\gamma_1 = E\left(\tilde{R}_M\right)$, the expected excess return on the market portfolio.

An alternative hypothesis of the pricing of capital assets arises from the relaxation of one of the assumptions of the traditional form of the capital asset pricing model. Relaxation of the assumption that riskless borrowing and lending opportunities are available leads to the formulation of the two-factor model. In equilibrium, the expected returns $E\left(\tilde{r}_j\right)$ on an asset will be given by

$$E\left(\tilde{R}_j\right) = E\left(\tilde{r}_Z\right) + \left[E\left(\tilde{r}_M\right) - E\left(\tilde{r}_Z\right)\right]\beta_j \tag{25}$$

where $E\left(\tilde{r}_Z\right)$ is the expected return on a portfolio that has a zero covariance (and thus $\beta_Z = 0$) with the return on the market portfolio $\tilde{r}_M$. In the context of this model, the return on 30-day Treasury Bills (which we have used as a proxy for a "riskless" rate) simply represents the return on a particular asset in the system. Thus, subtracting r_F from both sides of (25), we can rewrite (25) in terms of "excess" returns as

$$E\left(\tilde{R}_j\right) = \gamma_0 + \gamma_1 \beta_j \tag{26}$$

where $\gamma_0 = E\left(\tilde{R}_Z\right)$ and $\gamma_1 = E\left(\tilde{R}_M\right) - E\left(\tilde{R}_Z\right)$.

The traditional form of the asset pricing model implies that $\gamma_0 = 0$ and $\gamma_1 = E\left(\tilde{R}_M\right)$ and the two-factor model implies that $\gamma_0 = E\left(\tilde{R}_Z\right)$, which is not necessarily zero and that

$\gamma_1 = E(\tilde{R}_M) - E(\tilde{R}_Z)$. In addition, several other models arise from relaxing some of the assumptions of the traditional asset pricing model which imply $\gamma_0 \neq 0$ and $\gamma_1 \neq E(R_M)$. These models involve explicit consideration of the problems of measuring R_M, the existence of nonmarketable assets, and the existence of differential taxes on capital gains and dividends, and we shall briefly outline them. Our main emphasis has been to test the strict traditional form of the asset pricing model; that is, $\gamma_0 \neq 0$? We have made no attempt to provide direct tests of these other alternative hypotheses.

To test the traditional model, we used all securities listed on the New York Stock Exchange at any time in the interval between 1926 and 1966. The problem we faced was to obtain efficient estimates of the mean of the beta factor and its variance. It would be possible to test the alternative hypotheses by selecting one security at random and estimating its beta from the time series and ascertaining whether its mean return was significantly different from that predicted by the traditional form of the capital asset pricing model. However, this would be a very inefficient test procedure.

To gain efficiency, we grouped the securities into ten portfolios in such a way that the portfolios had a large spread in their β's. However, we knew that grouping the securities on the basis of their estimated β's would not give unbiased estimates of the portfolio "Beta," since the β's used to select the portfolios would contain measurement error. Such a procedure would introduce a selection bias into the tests. To eliminate this bias we used an instrumental variable, the previous period's estimated beta, to select a security's portfolio grouping for the next year. Using these procedures, we constructed ten portfolios whose estimated β's were unbiased estimates of the portfolio "Beta." We found that much of the sampling variability of the β's estimated for individual securities was eliminated by using the portfolio groupings. The β's of the portfolios constructed in this manner ranged from 0.49 to 1.5, and the estimates of the portfolio β's for the subperiods exhibited considerable stationarity.

The time series regressions of the portfolio excess returns on the market portfolio excess returns indicated that high-beta securities had significantly negative intercepts and low-beta securities had significantly positive intercepts, contrary to the predictions of the traditional form of the model. There was also considerable evidence that this effect became stronger through time, being strongest in the 1947-65 period. The cross-sectional plots of the mean excess returns on the portfolios against the estimated β's indicated that the relation between mean excess return and β was linear. However, the intercept and slope of the cross-sectional relation varied in different subperiods and were not consistent with the traditional form of the capital asset pricing model. In the two prewar 105-month subperiods examined, the slope was steeper in the first period than that predicted by the traditional form of the model, and it was flatter in the second period. In each of the two 105-month postwar periods it was considerably flatter than predicted. From the evidence of both the time series and cross-sectional runs, we were led to reject the hypothesis that γ_0 in (26) was equal to zero; we therefore concluded that the traditional form of the asset pricing model is not consistent with the data.

We also attempted to make explicit estimates of the time series of returns on the beta factor in order to obtain a more efficient estimate of its mean and variance and thereby enable ourselves to directly test whether or not the mean excess return on the beta factor was zero. We derived a minimum-variance, unbiased linear estimator of the returns on the β factor using our portfolio return data. We showed that, given the independence of the residuals the optimum estimator requires knowledge of the unobservable residual variances of each of the portfolios but that this problem could be avoided if they were equal. Under this assumption of equal residual variances, we estimated the time series of returns on the beta factor. However, if these assumptions (i.e., the independence of the residuals and equality of their variances) are not valid--and there is reason to believe they are not--more complicated procedures are necessary to obtain minimum-variance estimates. Such estimators, which use the complete covariance structure of the portfolio returns are available (although not derived here). However, we feel that a straightforward application of these procedures to the return data would result in the introduction of serious *ex post* bias in the estimates. Thus we have left a complete investigation of these problems, as well as more detailed tests of the two-factor model, to a future paper. In order to fully utilize the properties of the two-factor model in a number of applied problems (such as portfolio evaluation, see Jensen (1971) and various issues in valuation theory), it will be necessary to have minimum-variance unbiased estimates of the time series of returns on the beta factor, and we hope to provide such estimates in the not-too--distant future.

The evidence obtained from the time series of returns on the beta factor indicated that the beta factor had a nonzero mean and that the mean was non-stationary over time. It seems to us that we have established the presence and significance of the beta factor in explaining security returns but, as mentioned earlier, we have not provided any direct tests aimed at explaining the existence of the beta factor. We have, however, suggested an economic rationale for why capital market equilibrium is consistent with the finding of this second factor. Black (1970) has shown that if riskless borrowing opportunities are not available, the equilibrium expected returns on an asset will be a linear function of two factors, one the β factor, the other the market factor.

In addition, Black and Jensen (1970) have demonstrated that if assets are omitted from the estimated market return, a model similar in some ways to the two-factor model would result. (Roll's analysis (1969) is relevant to this issue as well.) That is, it yields a model similar in structure to (26) and implies that $\gamma_0 \neq 0$. However, it is clear from Figures 6a and 6b and Table 7 that the beta factor (the intercept in the figures and γ_0 in Table 7) is highly variable and any alternative hypothesis must be consistent with this phenomenon. In other words, it is not sufficient for an alternative model to simply imply a nonzero but constant intercept in (26).

Others have provided alternative models that are similar in structure to the Black-Jensen results. For example, Mayers (1972) has developed an equilibrium model incorporating the existence of nonmarketable assets and has shown that the basic linear relation of the traditional model is unaltered, but the constant term γ_0 will be nonzero and γ_1 will not equal $E(R_M)$. The implications of his model for the structure of asset returns are virtually identical to those of the omitted assets model. Brennan [1970] has derived the equilibrium structure of security returns when the effects of a differential tax on dividends and capital gains are considered. He also concludes that the basic linearity of the traditional model is unchanged, but a nonzero constant term must be included and γ_1 will not equal $E(R_M)$. Black and Scholes (1970), however, have tested for the existence of dividend effects and have found that the differential tax on dividends and capital gains does not affect the structure of security returns and hence cannot explain the results reported here.

There are undoubtedly other economic hypotheses that are consistent with the findings of the existence of a second factor and consistent also with capital market equilibrium. Each hypothesis must be tested directly to determine whether it can account for the presence of the β factor. The Black Scholes investigation of dividend effects is an example of such a test.

Appendix: The Grouping Solution to the Measurement Error Problem

Consider first the estimate $\hat{\beta}_j$ of the risk parameter in more detail. We will want to test (10) over some holding period, but we must first obtain the estimates of the risk parameter $\hat{\beta}_j$, from the time series equation given by (6). For simplicity, we shall assume that the $\tilde{e}_{jt}$ are independently distributed and have constant variance for all j and t. The least-squares estimate of β_j in (6), $\hat{\beta}_j$, is thus unbiased but subject to a sampling error $\in_j$ as in (7), and the variance of the sampling error of the estimate $\hat{\beta}_j$ is

$$var\left(\hat{\beta}_j \middle| \beta_j\right) = \sigma^2\left(\tilde{\in}_j\right) = \frac{\sigma^2\left(\tilde{e}_j\right)}{\phi} = \frac{\sigma^2\left(\tilde{e}\right)}{\phi} \tag{A.1}$$

since $\sigma^2\left(\tilde{e}_j\right)$ was assumed equal for all j, and where

$$\phi = \sum_{t=1}^{T}\left(R_{Mt} - R_M\right)^2 \tag{A.2}$$

is the sample sum of squared deviations of the independent variable over the T observations used in the time series estimating equation. Hence using (11) we see that

$$p\,lim\,\hat{\gamma} = \frac{\gamma_1}{1 + \sigma^2\left(\tilde{e}\right) / \phi S^2\left(\beta_j\right)} \tag{A.3}$$

Let us assume that we can order the firms on the basis of β_j or on the basis of some instrumental variable highly correlated with β_j but independent of $\tilde{\in}_j$. Given the N ordered firms, we group them into M equal-size contiguous subgroups, represented by $K = 1, 2, \ldots, M$ and calculate the average return for each group for each month t according to

$$\tilde{R}_K = \frac{1}{L}\sum_{j=1}^{L}\tilde{R}_{Kjt} \qquad K = 1, 2, \ldots, M \tag{A.4}$$

$$L = \frac{N}{M} \quad \text{(assumed to be integer)} \tag{A.5}$$

where $\tilde{R}_{Kjt}$ is the return for month t for security j in group K. We then estimate the systematic risk of the group by applying least squares to

$$\tilde{R}_{Kt} = \alpha_K + \beta_K \tilde{R}_{Mt} + \tilde{e}_{Kt} \quad \begin{cases} K = 1, 2, \ldots, M \\ t = 1, 2, \ldots, T \end{cases} \tag{A.6}$$

where

$$\tilde{e}_{Kt} = \frac{1}{L}\sum_{j=1}^{L}\tilde{e}_{Kjt} \tag{A.7}$$

and

$$\sigma^2(\tilde{e}_{Kt}) = \frac{\sigma^2(\tilde{e})}{L} \tag{A.8}$$

Equation (A.8) holds, since, by assumption, the $\tilde{e}_{Kjt}$ are independently distributed with equal variance. The least squares estimate of β_K in (A.6) is $\hat{\beta}_K = \beta_K + \tilde{\in}_K$ and its variance is

$$var\left(\hat{\beta}_K \middle| \beta_K\right) = \sigma^2(\tilde{\in}_K) = \frac{\sigma^2(\tilde{e})}{\phi L} \tag{A.9}$$

Now if we estimate the cross-sectional relation (10) using our M observations on $\overline{R}_K = \sum_{t=1}^{T} R_{Kt}/T$ and $\hat{\beta}_K$ for some holding period, we have

$$\overline{R}_K = \gamma_0 + \gamma_1 \hat{\beta}_K + \overset{\circ}{\overline{e}}_K \qquad \text{(A.10)}$$

where

$$\overset{\circ}{\overline{e}}_K = \sum_{t=1}^{T} \frac{\overset{\circ}{\overline{e}}_{Kt}}{T} = \overline{e}_K - \gamma_1 \tilde{\in}_K \qquad \text{(A.11)}$$

Now the large sample estimate of y, in (A.10)

$$p\,lim\,\hat{\gamma}_1 = \frac{\gamma_1}{1 + \dfrac{p\,lim\,\sigma^2(\tilde{\in}_K)}{p\,lim\,S^2(\beta_K)}} = \frac{\gamma_1}{1 + \dfrac{p\,lim\,\frac{1}{L}\sigma^2(\tilde{e})}{\phi S^2(\beta_K)}} = \gamma_1 \qquad \text{(A.12)}$$

since $p\,lim\,\sigma^2(\tilde{e})/L = 0$ as long as $L \to \infty$ as $N \to \infty$, and this is true as long as we hold the number of groups constant. Thus these grouping procedures will result in unbiased estimates of the parameters of (10) for large samples. Note that $S^2(\beta_K)$, the cross-sectional sample variance of the true group risk coefficients, is constant with increasing L so long as securities are assigned to groups on the basis of the ranked β_j. Note also, however, that if we randomly assigned securities to the *M* groups we would have $p\,lim\,S^2(\beta_K) = p\,lim\,S^2(\beta_j)/L$ and (A.12) would thus be identical to (A.3). Therefore, random grouping would be of no help in eliminating the bias. As can be seen, the grouping procedures we have already described in the time series tests accomplish these results. While we expect these procedures to substantially reduce the bias[16] they

16 As mentioned earlier, the choice of the number of groups is somewhat arbitrary and, for any given sample size, involves a tradeoff between the bias and the degree of sampling error in the estimates of the parameters in (10). In an unpublished study of the properties of the grouping procedures by simulation techniques, Jensen and Mendu Rao have found that, when $\sigma^2(\tilde{\in}_j) = S^2(\beta_j)$, the use of ten groups with a total sample size of N= 400, yields estimates of the coefficient γ_1 in (10) which, on the average, are biased

cannot completely eliminate it in our case because the $\tilde{e}_j$ and therefore the $\tilde{\in}_j$ are not independent across firms. However, as discussed in Section III, we expect the remaining bias to be trivially small.

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downward by less than 0.9% of their true value and have a standard error of estimate about 50% higher than that obtained with ungrouped data. The ungrouped sample estimates were, of course, 50% of their true values on the average [as implied by (11) for these assumed variances].

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